Math 621 Homework Assignment 1 Spring 2013

Due: Friday, February 1

1. Lang, Ch. 1 Section 2 page 11: 7, 10, 11, 12, 13

2. Lang, Ch. 1 Section 3 page 17 problem 4: Let $f(z) = e^z$. Describe the image under f of the following sets:

   (a) $\{z = x + iy \mid x \leq 1 \text{ and } 0 \leq y \leq \pi\}$.

   (b) $\{z = x + iy \mid 0 \leq y \leq \pi \ \ (\text{no condition on } \ x)\}$.

3. Lang, Ch. 1 Section 4 page 26 problem 3: Show that for any complex number $z \neq 1$, we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

   If $|z| < 1$, show that

$$\lim_{n \to \infty} (1 + z + \cdots + z^n) = \frac{1}{1 - z}$$

4. Let u, v be real valued functions defined on an open set U in $\mathbb{R}^2$. Prove that if u and v have continuous partial derivatives, then the function

$$\begin{aligned} f : U &\to \mathbb{R}^2 \\ (x, y) &\mapsto (u(x, y), v(x, y)) \end{aligned}$$

   is differentiable throughout U in the sense that

$$\lim_{(\Delta x, \Delta y) \mapsto (0,0)} \frac{\| f(x + \Delta x, y + \Delta y) - [f(x, y) + df(\Delta x, \Delta y)] \|}{\| (\Delta x, \Delta y) \|} = 0.$$

   Hint: Bound the quotient by a sum of two terms each depending only on u or v, define $a(t) := u(z + t \cdot \Delta z)$ and $b(t) := v(z + t \cdot \Delta z)$, $0 \leq t \leq 1$, and use the Mean Value Theorem for $a(t)$ and $b(t)$.

5. Let

$$f(z) = \begin{cases} (\bar{z}^2/z), & \text{if } \ z \neq 0, \\ 0, & \text{if } \ z = 0. \end{cases}$$

   Show that the real and imaginary parts of f satisfy the Cauchy-Riemann equations at $z = 0$. Is f holomorphic at $z = 0$?

6. Ahlfors, Ch. 1 Section 2.1 page 15 problem 2: Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if

$$a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_3 a_1.$$

7. (a) Show that if $A \neq 0$, the set of points

$$\{(x, y) \in \mathbb{R}^2 \ : \ A(x^2 + y^2) + Bx + Cy + D = 0$$

   is either empty or a circle. Determine the center and the radius. What happens when $A = 0$?

(b) Show that the set of points

$$\{z \in \mathbb{C} \ : \ \left|\frac{z-z_1}{z-z_2}\right| \ = \ K; \ K > 0, \ K \neq 1\}$$

is a circle. Determine the center and the Radius. What happens when $K = 1$?

8. Let $P(z) = \sum_{n=0}^{d} a_n z^n$, $a_d \neq 0$, be a polynomial of degree d. Show that there exist positive constants k, K, and R such that

$$k|z|^d \leq |P(z)| \leq K|z|^d, \quad \text{for} \quad |z| > R.$$

9. Given a complex valued function f of one complex variable z, define

$$\frac{\partial}{\partial z} f := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f, \qquad \frac{\partial}{\partial \bar{z}} f := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f.$$

Assume f is holomorphic.

(a) Show that $\frac{\partial}{\partial \bar{z}} f = 0$, $\frac{\partial}{\partial z} f = f'$, $\frac{\partial}{\partial \bar{z}}\left(\bar{f}\right) = \overline{\left(\frac{\partial f}{\partial z}\right)}$, and $\frac{\partial}{\partial z}\bar{f} = 0$.

(b) Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

(c) Use parts 9a and 9b to show that $\log|f(z)|$ is HARMONIC provided $f \neq 0$.