Math 621 Solution of the Midterm Spring 2006

1. (10 points) Find the power series expansion of $f(z) = \frac{1}{z+i}$, centered at $z = 1$. What is its radius of convergence?

   Answer: $\frac{1}{z+i} = \frac{1}{(z-1)+1+i} = \frac{1}{1+i}\frac{1}{\left(\frac{z-1}{1+i}\right)+1} = \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1+i}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-1)^n$. The radius of convergence is $|1+i| = \sqrt{2}$.

2. (18 points) Compute the following integrals. Justify all your steps.

   (a) $\displaystyle\int_C \frac{e^{2iz}+z^2+3}{(z-2)^3}dz$, where $C = \{z : |z| = 3\}$ transversed counterclockwise.

   Answer: Set $f(z) = e^{2iz} + z^2 + 3$. Then
   $$\int_C \frac{e^{2iz}+z^2+3}{(z-2)^3}dz = 2\pi i \cdot \frac{f^{(2)}(2)}{2!} = \pi i((-4)e^{4i}+2) = \pi[\sin(4) + i(2-4\cos(4))].$$

   (b) $I := \displaystyle\int_C \frac{dz}{z(1-z)}$, along the path C consisting of the line segments C_1, C_2, C_3, where C_1 goes from -2 to $-i$, C_2 goes from $-i$ to $1+i$, and C_3 goes from $1+i$ to 2.

   Answer: Write $\frac{1}{z(1-z)} = \frac{1}{z} - \frac{1}{z-1}$. We find primitives, for each summand, defined in an open set containing C, by setting

$$\begin{aligned} f(z) &= \log(z) \quad \text{with} \quad \arg(z) \in (-3\pi/2, \pi/2), \\ g(z) &= \log(z-1) \quad \text{with} \quad \arg(z-1) \in (-\pi/2, 3\pi/2). \end{aligned}$$

   Thus, $I = \int_C f'(z) - g'(z)dz = [f-g]_{-2}^{2} = \{\ln(2) - \ln(1)\} - \{(\ln(2) - i\pi) - (\ln(3) + i\pi)\} = \ln(3) + 2\pi i$.

3. (18 points) Determine if the following statements are true or false. If true, prove the statement. If false, give a counter example.

   (a) Let $f(z)$ and $g(z)$ be two entire functions, such that $|f(z)| \leq |g(z)|$, for every complex number z. Then $f(z) = cg(z)$, for some constant c.

   Answer: True. If f is identically 0, take $c = 0$. Otherwise, $g(z)$ is not identically zero. The function $h(z) := \frac{f(z)}{g(z)}$ has isolated singularities, since the zeros of $g(z)$ are isolated. Furthermore, $h(z)$ is bounded, by assumption. Hence, all the singularities of $h(z)$ are removeable and $h(z)$ extends to a bounded entire function, which must be constant, by Liouville's Theorem.

   (b) Let f and g be holomorphic functions on the upper-half-plane $\mathbb{H} := \{x + iy \ : \ y > 0\}$. If $f(\frac{i}{n}) = g(\frac{i}{n})$, for all integers $n \geq 1$, then $f = g$.

   Answer: False. Take $f(z) = 1$ and $g(z) = e^{2\pi/z}$.

   (c) Let $U \subset \mathbb{C}$ be an open connected subset. If both $f(z)$ and $\overline{f(z)}$ are holomorphic functions on U, then f is constant in U.

   Answer: True. Everybody did this part correctly.

4. (18 points) Find all linear fractional transformations, which map the upper half of the unit disk $U := \{z \ : \ |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ onto itself, and satisfy $f(1) = -1$. Justify your answer. *Hint:* There are infinitely many such maps.

Answer: f must map the boundary to the boundary preserving the orientation. Being a conformal map (preserving angles), f must also map the set of vertices $\{1,-1\}$ to itself in a one-to-one fashion. Hence, $f(-1)=1$. The orientation determines the cyclic order of the boundary points $\{-1,0,1\}$, which must hence be preserved. Consequently, $\lambda := f(0)$ must belong to the upper-half of the unit circle.

We can proceed to argue in two ways. We can prove that an automorphism f of U exists, for $\lambda = e^{i\theta}$, and for any choice of $\theta \in (0,\pi)$, and then compute it. Or we can compute f, for every such choice, and then prove that it indeed maps U onto itself.

First method: Let $W := \{x+iy \ : x>0, y>0\}$ be the first quadrant. The linear fractional transformation $h(z) = -\left(\frac{z+1}{z-1}\right)$, sending $(-1,0,1)$ to $(0,1,\infty)$, takes U onto W. The linear fractional transformation $\tau(z) = \frac{i}{z}$ maps W onto itself and interchanges the x and y axis. Now W is invariant under multiplication by any positive real number t. So the composition $f(z) = (h^{-1} \circ t\tau \circ h)(z) = \frac{(-1-ti)z+(-1+ti)}{(1-ti)z+(1+ti)}$ is the l.f.t we were looking for, with $\lambda = f(0) = \frac{-1+ti}{1+ti}$.

Second method: The l.f.t taking $(-1,0,1)$ to $(1,\lambda,-1)$ is given by $f(z) = \frac{z-\lambda}{\lambda z-1}$. It maps the real line to the unit circle, since there is a unique circle or line through three distinct points of the plane and $\{1,\lambda,-1\}$ are on the unit circle. Observe that $f^2 = id$. Hence, f takes the unit circle to the real line. It follows that f maps U either onto itself, or onto $\mathbb{CP}^1 \setminus \overline{U}$. We constructed f to preserve the orientation of the boundary (by making sure it preserves the cyclic order). Hence, f maps U onto itself. Alternatively, check that $f^{-1}(\infty) = \frac{1}{\lambda}$ does not belong to U. So $f(U) = U$.

5. (18 points) Let f be a holomorphic function on the closed disk $\{z \ : \ |z| \leq R\}$. Set $M := \max\{|f(z)| \ : \ |z| = R\}$. a) Prove the inequality

$$\left|f^{(n)}(a)\right| \quad \leq \quad \frac{n! \cdot MR}{(R-|a|)^{n+1}},$$

for any complex number a, such that $|a| < R$.

Answer: First Method: (Estimate the integral in Cauchy's Formula) We have

$$f^{(n)}(a) \quad = \quad \frac{n!}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-a)^{n+1}} dz.$$

Thus, $\left|f^{(n)}(a)\right| \ \leq \ \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{|z-a|^{n+1}} \left|iRe^{i\theta}\right| d\theta \ \leq \ \frac{n!R}{2\pi} \int_0^{2\pi} \frac{M}{(R-|a|)^{n+1}} d\theta \ = \ \frac{n!MR}{(R-|a|)^{n+1}}$

Second method: Use Cauchy's estimate with the circle C_a of radius $R-|a|$ centered at a. f is holomorphic on that disk, since that disk is contained in the disk of radius R centered at 0, We need however a bound for $|f|$ on the circle C_a. By the *Maximum principle*, we can take M as an upper bound (we did not cover the Maximum principle before the exam, so some students used this method without explaining this point). We get

$$\left|f^{(n)}(a)\right| \quad \leq \quad \frac{n!M}{(R-|a|)^n} \quad < \quad \frac{n!MR}{(R-|a|)^{n+1}}.$$

b) What can you deduce about the radius ρ of convergence of the Taylor series of f centered at a?

Answer: First method: $\frac{1}{\rho} = \limsup_{n\to\infty} \left|\frac{f^{(n)}(a)}{n!}\right|^{1/n} \leq \lim_{n\to\infty} \frac{1}{R-|a|}\left(\frac{MR}{R-|a|}\right)^{1/n} = \frac{1}{R-|a|}$. Hence, $\rho \geq R - |a|$.

Second method: The same conclusion is obtained via Taylor's Theorem and the observation that f is holomorphic in the disk of radius $R - |a|$ centered at a.

6. (18 points) Let C_R be the circle of radius R centered at 0, oriented counterclockwise, and $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ a polynomial of degree $n \geq 2$.

   (a) Prove that $\displaystyle\lim_{R\to\infty}\int_{C_R}\frac{dz}{P(z)} = 0$.

   Answer: Observe, that $\lim_{z\to\infty}\left|\frac{P(z)}{z^n}\right| = 1$. Hence, there exists $R_0 > 0$, such that $|P(z)| > \frac{1}{2}|z|^n$, for all z satisfying $|z| > R_0$. Now estimate for $R > R_0$ the integral:

   $\left|\int_{C_R}\frac{dz}{P(z)}\right| \leq \int_{C_R}\frac{|dz|}{|P(z)|} \leq \int_{C_R}\frac{|dz|}{\frac{1}{2}|z|^n} = \int_{C_R}\frac{|dz|}{\frac{1}{2}R^n} = \frac{4\pi}{R^{n-1}} \stackrel{R\to\infty}{\longrightarrow} 0,$

   since $n \geq 2$.

   (b) Assume, that the zeros z_i of $P(z)$ all satisfy $|z_i| < R$. Prove that $\displaystyle\int_{C_R}\frac{dz}{P(z)} = 0$.

Answer: The main idea is to prove the equality $\int_{C_{R'}}\frac{dz}{P(z)} = \int_{C_R}\frac{dz}{P(z)}$, for any $R' > R$, and then use part 6a.

Method 1: The equality follows easily from the General Form of Cauchy's Theorem, since the circles C_R and $C_{R'}$ are homologous in the open set $\{z \,:\, |z| > R\}$ where $\frac{1}{P(z)}$ is holomorphic. However, we did not cover the General Form of Cauchy's Theorem before the exam. If you did part 6a correctly, you got the full credit. (Several students attempted to prove part 6b first).

Method 2: (without the General Form of Cauchy's Theorem) Using partial fractions, and the Fundamental Theorem of Algebra, we have

$$\frac{1}{P(z)} = \sum_i\sum_{j=1}^{m_i}\frac{A_{i,j}}{(z-z_i)^j},$$

where $A_{i,j}$ are constants, and m_i is the multiplicity of z_i as a root of P. Cauchy's formula for higher derivatives (applied to the constant function 1), yields, for $R' \geq R$, the second equality below:

$$\int_{C_{R'}}\frac{dz}{P(z)} = \sum_i\sum_{j=1}^{m_i}A_{i,j}\int_{C_{R'}}\frac{dz}{(z-z_i)^j} = (2\pi i)\sum_i A_{i,1}.$$

Hence, the integral is independent of R'.