Math 621

1. Compute the following integrals:

(a)
$$\int_{0}^{\pi/2} \frac{dx}{a + \sin^{2}(x)}, \quad |a| > 1,$$

(b) $\int_{0}^{\infty} \frac{x^{2} dx}{x^{4} + 5x^{2} + 6},$
(c) $\int_{0}^{\infty} \frac{\cos(x)}{x^{2} + a^{2}} dx, \quad a > 0 \text{ real},$
(d) $\int_{0}^{\infty} \frac{x \sin(x)}{x^{2} + a^{2}} dx, \quad a \ge 0, \text{ real},$
(e) $\int_{0}^{\infty} \frac{x^{1/3}}{1 + x^{2}} dx$

- 2. (a) Basic Exam September 98 Problem 9b: Prove that the image of the complex plane \mathbb{C} , under a non-constant entire function, is dense in \mathbb{C} .
 - (b) Prove that there does not exist a one-to-one conformal map from the complex plane \mathbb{C} onto the unit disk.
- 3. Lang page 307 Problem 7: The holomorphic automorphism group of a simply connected open set U acts transitively on points. More precisely, let $U \subset \mathbb{C}$ be a simply connected open set, z_1 , z_2 two points in U. Use the Riemann-Mapping-Theorem (Lang, page 306) to prove that there exists a holomorphic automorphism f of U such that $f(z_1) = z_2$. Distinguish the cases when $U = \mathbb{C}$ and $U \neq \mathbb{C}$.
- 4. Basic Exam, January 99 Problem 7: Find a one-to-one conformal map from the region obtained as the intersection of the two unit disks:

$$\{z : |z| < 1\} \cap \{z : |z - 1| < 1\}$$

onto the upper half plane. Hint: show that the angle between the two circles, at each of the two points of intersection, is $2\pi/3$. For the smoothing of the boundary, observe that the function $z^{1/\alpha}$, $0 < \alpha \leq 2$, is well defined on the region $\Omega := \{z : Arg(z) \in (0, \alpha \cdot \pi)\}$ (why?) and maps Ω onto the upper half plane.

Harmonic functions and boundary value problems: A boundary value problem, or Dirichlet problem is the problem of finding a harmonic function with given boundary values. For example, the function $\arg(z)$, with values in $[0, \pi]$, is the unique harmonic function h in the upper-half-plane, satisfying the boundary problem $h(x, 0) = \begin{cases} 0 & \text{if } x > 0 \\ \pi & \text{if } x < 0 \end{cases}$ (and with a jump singularity at the boundary point (0, 0)). There is a nice integral formula (the Poisson formula), for recovering the harmonic function given its boudary values (Ch 6 is Ahlforse, Ch VIII sec 4 in Lang). The problems below can all be solved without it, using the techniques of fractional linear transformations studied in this course, combined with the examples of harmonic functions we encountered as real and imaginary parts of elementary holomorphic functions. 5. (a) Find a function Φ , harmonic on the domain $D := \{z : 1 < |z| < 5\}$ and with the following boundary values:

$$\Phi(x,y) = \begin{cases} 0, & \text{if } |z| = 1, \\ 1, & \text{if } |z| = 5. \end{cases}$$

- (b) Find the level curves of Φ .
- 6. Ahlfors, page 171 Problem 4: Let C_1 , C_2 be complementary arcs on the unit circle. Set U = 1 on C_1 and U = 0 on C_2 .
 - (a) Find explicitly a harmonic function P_U in the open unit disk satisfying

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(e^{i\theta_0})$$

provided $U(e^{i\theta})$ is continuous at $e^{i\theta_0}$.

- (b) Show that $2\pi P_U(z)$ equals the length of the arc, opposite C_1 , cut off by the straight lines through z and the end points of C_1 .
- 7. (a) Basic Exam, August 97 Problem 9: Find a function u, harmonic in the unit disk, continuous on $\{z : |z| \le 1\} \setminus \{1, i\}$, and satisfying

$$\left\{ \begin{array}{ll} u = 0 & {\rm on} \; \{e^{i\theta} \; : \; 0 < \theta < \pi/2 \} \\ u = 1 & {\rm on} \; \{e^{i\theta} \; : \; \pi/2 < \theta < 2\pi \} \end{array} \right.$$

- (b) Find the level curves of u.
- (c) Find a harmonic conjugate v of u.
- (d) How are the level sets of u and v related?
- 8. (a) Solve the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega := \{ |z| < R \text{ and } Im(z) > 0 \} \\ u = 0 & \text{on } \{ |z| < R \text{ and } Im(z) = 0 \} \\ u = 1 & \text{on } \{ |z| = R \text{ and } Im(z) > 0 \} \end{cases}$$

in two ways:

- i. By using a conformal map from Ω onto the first quadrant of the complex plane.
- ii. By using the Reflection Principle (Lang Theorems 1.1 and 1.2 page 294) and your solution to Problem 6.
- (b) Find the level sets of u.