Math 621 Homework Assignment 7 Spring 2006

**Due: Wednessday, May 3**

1. Let $P(z)$ and $Q(z)$ be polynomials and suppose that $\deg Q \geq \deg P + 2$. Show that
$$\sum_{\zeta} Res_{\zeta}(P/Q) = 0,$$
where the sum runs over all singularities of the rational function $P/Q$. Do this problem in two ways: (1) Directly, without considering the residue at $\infty$. (2) As a special case of problem 3.

2. Lang Problem 37 page 190: Let $f$ be analytic on $\mathbb{C}$ with exception of a finite number of isolated singularities $\{z_1, \ldots z_n\}$. Define the *residue at infinity*
$$Res_{\infty} f(z)dz \ := \ -\frac{1}{2\pi i}\int_{|z|=R} f(z)dz$$
for $R > \max\{|z_1|, \ldots |z_n|\}$.

   (a) Show that $Res_{\infty} f(z)dz$ is independent of $R$.

   (b) Show that the sum of the residues of $f$ in the extended complex plane $\mathbb{CP}^1$ is equal to zero. (This result is often refered to as *The residue Theorem.*)

3. Lang Problem 38 page 190 (Cauchy's Residue Formula on the Riemann Sphere).

4. Basic Exam, September 1998 Problem 5: Compute $\displaystyle\int_C \frac{z^4 e^{1/z}}{1-z^4}dz$ where $C$ denotes the circle $\{|z| = 2\}$ transversed counterclockwise. *Hint: Use the residue at infinity (problem 3) to save computations.*

5. Basic Exam, January 2000 Problem 7: Let $f$ be a one-to-one holomorphic map from a region $D_1$ onto a region $D_2$. Suppose that $D_1$ contains the closure of the disk $\Delta := \{|z - z_0| < \rho\}$. Prove that for $w \in f(\Delta)$ the inverse function $f^{-1}(w)$ is given by
$$f^{-1}(w) \ = \ \frac{1}{2\pi i}\int_{\{|z-z_0|=\rho\}} \frac{f'(z)}{f(z)-w} \cdot z dz$$

6. Let $P(z)$ be a polynomial of degree $d$ and assume that the roots $\zeta_1, \ldots, \zeta_d$ of $P$ are simple. Show that for $R$ sufficiently large:
$$\int_{\{|z|=R\}} \frac{z^k P'(z)}{P(z)}dz \ = \ \sum_{i=1}^{d} \zeta_i^k.$$

7. Lang page 189 Problem 29: Let $U$ be a connected open set, and let $D$ be an open disk whose closure is contained in $U$. Let $f$ be analytic on $U$ and not constant. Assume that the absolute value $|f|$ is constant on the boundary of $D$. Prove that $f$ has at least one zero in $D$. *Hint: Consider $g(z) := f(z) - f(z_0)$ with $z_0 \in D$.*

8. Basic Exam January 2000 Problem 9: Prove that the equation
$$20\frac{z^3}{z^2+4} \ = \ e^z$$
has 3 roots in the unit disk $\{|z| < 1\}$.

9. Basic Exam, August 99 Problem 4: Let $\lambda$ be a real number larger that 1. Show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \mathrm{Re}(z) > 0\}$. Moreover, the solution is real.

10. Basic Exam, January 99 problem 3:

    (a) Determine the number of zeroes of $z^5 - 2z^2 + z + 1$ in the disk $\{z : \ |z| < 10\}$.

    (b) Compute the integral $\displaystyle\int_{\{z:|z|<10\}} \frac{3z^4+1}{z^5-2z^2+z+1} dz$.

11. (Recommended Problem, you need not handed it in) Lang Page 191 Problem 40 (modified): Let $a$, $b \in \mathbb{C}$ with $|a|$ and $|b| < R$. Let $C_R$ be the circle of radius $R$.

    a) Show that there is an analytic branch of $\sqrt{(z-a)(z-b)}$ defined in $\{z \ : \ |z| > R\}$ and such that $\frac{z}{\sqrt{(z-a)(z-b)}}$ is analytic with value 1 at $\infty$.

    b) Evaluate
$$\int_{C_R} \frac{zdz}{\sqrt{(z-a)(z-b)}}.$$

    c) Evaluate
$$\int_{C_R} \frac{dz}{\sqrt{(z-a)(z-b)}}.$$