Math 621 Homework Assignment 6 Spring 2006

Due: Monday, April 24

1. Ahlfors, page 136 Problem 6: If γ is a path, piecewise of type C^1, contained in the open unit disc D, then the integral

$$\int_\gamma \frac{|dz|}{1-|z|^2}$$

is called the *non-euclidean length* or *hyperbolic length* of γ. Let $f : D \to D$ be an analytic function from the disc into itself. Show that f maps every γ on a path with smaller or equal non-euclidean length. Deduce that a linear fractional transformation from D onto itself preserves non-euclidean lengths.

2. Ahlfors, page 136 Problem 7: (Modified) It can be shown, that the path of smallest non-euclidean length, joining the origin 0 to a point $z \in D$, is the straight line segment between them.

   (a) Use this fact to show that the path of smallest non-euclidean length, that joins two given points in the unit disk, is the piece of the circle C which is orthogonal to the unit circle ∂D. The shortest non-euclidean length is called the *non-euclidean distance.*

   (b) Show that the non-euclidean distance between z_1 and z_2 is

$$\frac{1}{2}\log\frac{1+\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}$$

3. (a) Show that single valued analytic branches of $f(z) = z^\alpha$, $\alpha \in \mathbb{C}$, and $f(z) = z^z$ can be defined in any simply connected region, which does not contain the origin.

   (b) (Ahlfors, problem 5 page 148) Show that a single vlaued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points 1 and -1 are in the same connected component of the complement. What are the possible values of $\int \frac{dz}{\sqrt{1-z^2}}$ over a closed curve in the region?

4. Laurent Serries: Lang page 164: 8, 12, 13

5. Problem 5 from the basic exam of August 99: Consider the Laurent series $\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, which is valid in the annulus $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$. Find the coefficients a_n with index $-\infty < n \leq -1$. Hint: Use integration.

6. Isolated Singularities: Lang page 170: 1a,c,e, 4

7. Find the value of the integral $\displaystyle\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$ for:

   (a) C the circle $|z-2| = 2$.

   (b) C the circle $|z| = 4$.

8. Suppose that $f(z) = \frac{g(z)}{h(z)}$, $g(z_0) \neq 0$ and $h(z)$ has a zero of order 2 at z_0. Prove that

$$Res_{z_0} f \;=\; \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h'''(z_0)}{3(h''(z_0))^2}$$

9. Compute the integral of the following functions over the circle $|z| = 2$:

   (a) $f(z) = \dfrac{1}{(z-3)(1+2z)^2(1-3z)^3}$

   (b) $f(z) = \dfrac{z^2}{1-e^{z/4}}$

   (c) $f(z) = \dfrac{\cos(1/z)}{1+z^4}$