Math 621 Homework Assignment 5 Spring 2006

Due: Monday, April 10

1. (a) Ahlfors, page 130 Problem 2: Show that a function which is analytic in the whole plane and has a non-essential singularity at ∞ reduces to a polynomial. (You may use Problem 7 in Homework assignment 4).

   (b) Lang, page 171 Problem 10: Show that any function, which is meromorphic on the extended complex plane, is a rational function.

2. (a) Show that the functions $\cos(z)$ and $\sin(z)$ have essential singularities at ∞.

   (b) Let $f(z) = \cos\left(\frac{1+z}{1-z}\right)$, $|z| < 1$. Find the set Z_f of zeroes of f. Does Z_f have any accumulation points? Explain. (See Lang, page 21 for the definition of an *accumulation point*).

3. Lang, page 171 Problem 11: Define the order $\mathrm{Ord}_p f$ of a meromorphic function f at a point p to be $\mathrm{Ord}_p f := \begin{cases} m & \text{if } p \text{ is a zero of } f \text{ of order } m \\ -m & \text{if } p \text{ is a pole of } f \text{ of order } m \end{cases}$

   Above, m could be zero, meaning that f is analytic at p and $f(p) \neq 0$.

   Let f be a meromorphic function on the extended complex plane $\mathbb{C}P^1$ (so a rational function by problem 1a).

   (a) Prove that $\displaystyle\sum_{p\in\mathbb{CP}^1} \mathrm{Ord}_p f = 0.$

   In other words, the number of points in the fiber $f^{-1}(0)$, counted with multiplicity, is equal to the number of points in $f^{-1}(\infty)$, counted with multiplicity.

   (b) Prove that all fibers $f^{-1}(\lambda)$, $\lambda \in \mathbb{CP}^1$, of f consist of the same number of points, provided they are counted with multiplicity,

4. Ahlfors, page 130 Problem 5: Let z_0 be an isolated singularity of an analytic function f. Prove that if $\mathrm{Re}(f)$ is bounded from above or below, then z_0 is a removable singularity. *Ahlfors' Hint:* Apply a linear l.f.t. *Note:* Personally, I find it easier to avoid using a l.f.t (which does not seem to help rule-out the case of a pole). Instead, a short proof can be obtained using both the Casorati-Weirstrass and the Open Mapping Theorems.

5. Let $\tau \in \mathbb{C}$ be a complex number and assume that $\mathrm{Im}(\tau) \neq 0$. A function f is said to be *doubly periodic with periods* 1 *and* τ if

   $$f(z+1) = f(z) \quad \text{and} \quad f(z+\tau) = f(z), \quad \text{for all } z \in \mathbb{C}.$$

   Show that every entire function, which is doubly periodic with periods 1 and τ, is necessarily constant. (We will see that there exist non-constant, doubly periodic, meromorphic functions $f : \mathbb{C} \to \mathbb{CP}^1$).

6. Jan 96 Basic Exam, Problem 5: Find the maximum value of the function $g(z) = |z^3 - z|$ on the disk $|z| \leq 2$. Justify your answer!

7. Lang page 213 Problem 1: Let f be analytic on the unit disc D, and assume that $|f(z)| < 1$ on the disc. Prove that if there exist two distinct points a, b in the disc, which are fixed under f (that is $f(a) = a$ and $f(b) = b$), then $f(z) = z$.

8. Lang, page 219 problem 8: Use Schwarz's Lemma to prove that $PSL(2, \mathbb{R})$ is the group $\mathrm{Aut}(\mathbb{H})$ of holomorphic automorphisms of the upper half plane. ($PSL(2, \mathbb{R})$ is naturally identified with the group of fractional linear transformations which are associated to invertible 2×2 matrices with *real* coefficients and determinant 1). *Hint:* (a) Show that $PSL(2, \mathbb{R})$ is an index 2 subgroup of $PGL(2, \mathbb{R})$. (b) Use Schwarz's Lemma to show that if f belongs to $\mathrm{Aut}(\mathbb{H})$, then f is a linear fractional transformation. (c) Show that if f belongs to $\mathrm{Aut}(\mathbb{H})$, then it belongs to $PGL(2, \mathbb{R})$. (d) Show that if f belongs to $PGL(2, \mathbb{R})$, then f map $\mathbb{H}$ either to $\mathbb{H}$ or to the lower-half-plane. (f) Show that if f belongs to $PGL(2, \mathbb{R})$, then $f(\mathbb{H}) = \mathbb{H}$, if and only if f belongs to $PSL(2, \mathbb{R})$ (calculate $f'(x)$, for $x \in \mathbb{R}$).

9. Lang page 213 Problem 2: Let $f : D \to D$ be a holomorphic map from the disc into itself. Prove that, for all $a \in D$, we have

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

Moreover, *equality for some a implies that f is a linear fractional transformation. Hint:* Let g be an automorphism of D such that $g(0) = a$, and let h be the automorphism which maps $f(a)$ on 0. Let $F = h \circ f \circ g$. Compute $F'(0)$ and apply the Schwarz Lemma.

10. Ahlfors, page 136 Problem 2: Let $f(z)$ be analytic and $\mathrm{Im}(f(z)) \geq 0$ for all z in the upper half plane $\mathbb{H}$. Show that for $z, z_0 \in \mathbb{H}$,

$$\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \frac{|z - z_0|}{|z - \bar{z}_0|}$$

and, writing $z = x + iy$,

$$\frac{|f'(z)|}{\mathrm{Im} f(z)} \leq \frac{1}{y}.$$

Moreover, equality, in either one of the two inequalities above, implies that f is a linear fractional transformation.