Math 621 Homework Assignment 4 Spring 2006

Due: Monday, March 27

1. Use Green's Theorem to prove a *weaker* version of Cauchy-Goursat's Theorem for a rectangle:

   Let f be a holomorphic function defined and having a **continuous** *derivative f' in an open set U containing a rectangle R. Then*
   $$\int_{\partial R} f dz \;=\; 0.$$
   Recall the statement of Green's Theorem: Let γ be an oriented piecewise smooth *simple* path (i.e., each connected component of γ does not intersect itself) in the plane. Assume that γ bounds a region D (and has the *induced orientation*, i.e., each smooth piece of γ is oriented so that D is on the left as you move along γ). Let $p(x,y)$, $q(x,y)$ be two functions which are defined and have **continuous partial derivatives** in an open set $U \subset \mathbb{R}^2$ containing D and γ. Then
   $$\int_\gamma pdx + qdy \;=\; \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy.$$

2. (a) Let D be an open disk in $\mathbb{C}$ and let f be continuous in D. Suppose that $\displaystyle\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in D. Prove that f is holomorphic.

   (b) Suppose that f is continuous in all of $\mathbb{C}$ and holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Prove that f is holomorphic everywhere.

3. Let U be an open subset of $\mathbb{C}$ and f_n a sequence of holomorphic functions which converges, uniformly on compact subsets of U, to a function f. Prove that f is holomorphic in U and that f_n' converges, uniformly on compact subsets of U, to f'.

4. Lang page 132 Problem 1: Find the integrals over the unit circle C:

   (a) $\displaystyle\int_C \frac{\cos(z)}{z} dz$ (b) $\displaystyle\int_C \frac{\sin(z)}{z} dz$ (c) $\displaystyle\int_C \frac{\cos(z^2)}{z} dz$

5. Ahlfors page 120 Problem 3: Compute $\displaystyle\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq \rho$.

   Hint: Make use of the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\frac{dz}{z}$.

6. Show that the successive derivatives of an analytic function at a point can never satisfy $\left|f^{(n)}(z)\right| > n!n^n$ in two ways: (a) Using Cauchy's Estimate. (b) Using Taylor's Theorem.

7. Lang page 132 Problem 3 (modified): Let f be an entire function, k a positive integer, and let $\| f \|_R$ be the maximum of $|f|$ on the circle of radius R centered at the origin. Then f is a polynomial of degree $\leq k$ if and only if there exist constants C and $R_0 \geq 0$ such that
   $$\| f \|_R \;\leq\; CR^k,$$
   for all $R \geq R_0$. (Note: one direction was proven in HW 1 Problem 8).

8. Lang page 159 Problem 7: Let f be analytic on a closed disc $\overline{D}$ of radius $b > 0$, centered at z_0. Show that

$$\frac{1}{\pi b^2} \int \int_D f(x+iy)dydx = f(z_0).$$

Hint: Use polar coordinates and Cauchy's Formula.

9. Lang page 159 Problem 9 (modified): Let f be analytic and $1:1$ on the unit disk D, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor series expansion of f. Show that

$$\text{area} f(D) \;=\; \pi \sum_{n=0}^{\infty} n|a_n|^2.$$