Math 621 Homework Assignment 2 Spring 2006

Due: Friday, February 24

1. (a) Let D_1 and D_2 be open connected sets in $\mathbb{C}$ and $f : D_1 \to D_2$ a holomorphic map. Show that if H is harmonic in D_2, then the composition $H \circ f$ is harmonic in D_1.

   (b) Let $D \subset \mathbb{R}^2$ be a connected open set and $u(x, y)$, $v(x, y)$ harmonic in D. Prove or disprove the following statements:

   i. The function $w(x, y) = e^{u(x,y)}$ is harmonic in D.

   ii. The function $w(x, y) = u(x, y)v(x, y)$ is harmonic in D.

   iii. If v is a harmonic conjugate of u, the function $w(x, y) = u^2(x, y) - v^2(x, y)$ is harmonic in D.

2. Ahlfors page 28 problem 3: Find the most general harmonic homogeneous polynomial of degree 3 (of the form $ax^3 + bx^2y + cxy^2 + dy^3$). Determine, by integration, the conjugate harmonic function and the corresponding analytic function (up to a constant).

3. Ahlfors page 28 problem 4: Show that if f is a holomorphic function with a constant absolute value $|f(z)|$, then f itself is a constant function.

4. Lang page 58 problem 4a, c, d, g, h

5. Lang page 59 problem 10.

6. Ahlfors, page 41 problem 8: For what values of z is $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ convergent? (Describe the set geometrically).

7. Lang page 26 problem 7. *Hint:* Show first the following identity

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{(1-z)^2}\left[\frac{z^{n-1}+z^{n-2}+\cdots+1}{z^n+z^{n-1}+\cdots+1} - \frac{z^{n-2}+z^{n-3}+\cdots+1}{z^{n-1}+z^{n-2}+\cdots+1}\right]$$

8. Ahlfors, page 47 problem 8: Express arc $\tan(w)$ in terms of the logarithm.

9. Ahlfors, page 47 problem 9: Use an appropriate branch of $\log(z)$ to define the angles of a triangle with vertices z_1, z_2, z_3, bearing in mind that the angles should be between 0 and π. With this definition, prove that the sum of the angles is π.

10. Lang Ch. II Sec 3 page 68 problem 4.

11. Find a fractional linear transformation that maps

    (a) $0, 1, \infty$ to $1, -1, 0$,

    (b) $0, i, -i$ to $1, -1, 0$.

12. Let $T(z) = \dfrac{z-i}{z+i}$. Determine the image of horizontal lines $Im(z) = b$ under T. When the image is a circle, determine the center and radius (you may find it helpful to use the notion of *symmetry*).

Review point set topology in $\mathbb{R}^n$ by reading

1. Lang, Ch I Section 4 pages 17-26 and

2. the Appendix "Connectedness" page 92-93 in Lang.