Math 621 Homework Assignment 1 Spring 2006 Due: Friday, February 10

- 1. Lang, Ch. 1 Section 2 page 11: 7, 10, 11, 12, 13
- 2. Lang, Ch. 1 Section 3 page 17 problem 4: Let $f(z) = e^{z}$. Describe the image under f of the following sets:
 - (a) $\{z = x + iy \mid x \le 1 \text{ and } 0 \le y \le \pi\}.$
 - (b) $\{z = x + iy \mid 0 \le y \le \pi \pmod{x}\}$.
- 3. Lang, Ch. 1 Section 4 page 26 problem 3: Show that for any complex number $z \neq 1$, we have

$$1 + z + \dots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

If |z| < 1, show that

$$\lim_{n \to \infty} (1 + z + \dots + z^n) = \frac{1}{1 - z}$$

4. Let u, v be real valued functions defined on an open set U in \mathbb{R}^2 . Prove that if u and v have continuous partial derivatives, then the function

$$\begin{array}{rccc} f: U & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (u(x,y),v(x,y)) \end{array}$$

is differentiable throughout U in the sense that

$$\lim_{(\Delta x, \Delta y) \mapsto (0,0)} \frac{\parallel f(x + \Delta x, y + \Delta y) - [f(x, y) + df(\Delta x, \Delta y)] \parallel}{\parallel (\Delta x, \Delta y) \parallel} = 0.$$

Hint: Bound the quotient by a sum of two terms each depending only on u or v, define $a(t) := u(z + t \cdot \Delta z)$ and $b(t) := v(z + t \cdot \Delta z)$, $0 \le t \le 1$, and use the Mean Value Theorem for a(t) and b(t).

5. Let

$$f(z) = \begin{cases} (\bar{z}^2/z), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the real and imaginary parts of f satisfy the Cauchy-Riemann equations at z = 0. Is f holomorphic at z = 0?

6. Ahlfors, Ch. 1 Section 2.1 page 15 problem 2: Prove that the points a_1 , a_2 , a_3 are vertices of an equilateral triangle if and only if

$$a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1.$$

7. (a) Show that if $A \neq 0$, the set of points

$$\{(x,y) \in \mathbb{R}^2 : A(x^2 + y^2) + Bx + Cy + D = 0$$

is either empty or a circle. Determine the center and the radius. What happens when A = 0?

(b) Show that the set of points

$$\{z \in \mathbb{C} : \left| \frac{z - z_1}{z - z_2} \right| = K; K > 0, K \neq 1\}$$

is a circle. Determine the center and the Radius. What happens when K = 1?

8. Let $P(z) = \sum_{n=0}^{d} a_n z^n$, $a_d \neq 0$, be a polynomial of degree d. Show that there exist positive constants k, K, and R such that

$$k|z|^{d} \le |P(z)| \le K|z|^{d}$$
, for $|z| > R$.

9. Given a complex valued function f of one complex variable z, define

$$\frac{\partial}{\partial z}f := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f, \quad \frac{\partial}{\partial \bar{z}}f := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f.$$

Assume f is holomorphic.

- (a) Show that $\frac{\partial}{\partial \bar{z}}f = 0$, $\frac{\partial}{\partial z}f = f'$, $\frac{\partial}{\partial \bar{z}}(\bar{f}) = \overline{\left(\frac{\partial f}{\partial z}\right)}$, and $\frac{\partial}{\partial z}\bar{f} = 0$.
- (b) Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$.
- (c) Use parts 9a and 9b to show that $\log |f(z)|$ is HARMONIC provided $f \neq 0$.