CYCLES ON ABELIAN $2n$-FOLDS OF WEIL TYPE FROM SECANT SHEAVES ON ABELIAN n-FOLDS

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Abstract. Let X be an abelian n-fold, $n \geq 2$, and $\hat{X}$ its dual abelian n-fold. Endow $V{=}H^1(X,\mathbb{Z}){\oplus}H^1(\hat{X},\mathbb{Z})$ with the natural symmetric bilinear pairing. The even cohomology $H^{ev}(X,\mathbb{Z})$ is the half spin representation of $\mathrm{Spin}(V)$. Fix an integer $d>0$ and set $K{:=}\mathbb{Q}(\sqrt{-d})$. A coherent sheaf F on X is a *K-secant sheaf*, if $ch(F)$ belongs to a 2-dimensional non-isotropic subspace P of $H^{ev}(X,\mathbb{Q})$ spanned by Hodge classes, such that the line $\mathbb{P}(P)$ intersects the spinorial variety in $\mathbb{P}[H^{ev}(X,K)]$ along two distinct complex conjugate points defined over K. The K-secant P determines an embedding $\eta : K \to \mathrm{End}_{\mathbb{Q}}(X{\times}\hat{X})$ and a rational $(1,1)$-form h on $X{\times}\hat{X}$. The triple $(X{\times}\hat{X},\eta,h)$ is a polarized abelian variety of Weil type, for a non-empty open subset of such K-secants.

Let F_1, F_2 be non-zero coherent sheaves on X with $ch(F_i)$ in such a K-secant P. Orlov constructed a derived equivalence $\Phi : D^b(X{\times}X) \to D^b(X{\times}\hat{X})$ categorifying the isomorphism of two $\mathrm{Spin}(V)$-representations, the tensor square $H^*(X{\times}X,\mathbb{Z})$ of the spin representation, and $\wedge^*V$. Assume that the rank of $E := \Phi(F_1^\vee \boxtimes F_2)$ is non-zero. We prove that the characteristic class $\exp\left(-\frac{c_1(E)}{\mathrm{rank}(E)}\right)ch(E)$ remains of Hodge type under every deformation of $(X{\times}\hat{X},\eta,h)$ as a polarized abelian variety of Weil type. The algebraicity of the Hodge-Weil classes of a deformation of the triple would follow, if E deforms as well as a possibly twisted object.

When $n=3$ we construct for every positive integer d an example, where $K=\mathbb{Q}(\sqrt{-d})$ and the derived dual E of $\Phi(F_1^\vee \boxtimes F_2)[1]$ is a simple reflexive sheaf over $X{\times}\hat{X}$. We prove that E deforms with $(X{\times}\hat{X},\eta,h)$ locally in the 9-dimensional moduli space of polarized abelian 6-folds of Weil type of discriminant -1, using the Semiregularity theorem of Buchweitz-Flenner.

The Hodge Conjecture for abelian fourfolds is known to follow from the above result.

Contents

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1. Introduction

1.1. **Abelian varieties of Weil-type.** A $2n$-dimensional abelian variety A is of *Weil-type*, if it admits an embedding $\eta : K \rightarrow \mathrm{End}_{\mathbb{Q}}(A)$, where $K = \mathbb{Q}(\sqrt{-d})$ for some positive integer d, such that each of the eigenspaces W and $\bar{W}$ of $\eta(\sqrt{-d})$, with eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$, intersects $H^{1,0}(A)$ in an n-dimensional subspace (see [W]). In that case $\wedge^{2n}W \oplus \wedge^{2n}\bar{W}$ is a $Gal(K/\mathbb{Q})$-invariant 2-dimensional subspace of $H^{2n}(A, K)$, corresponding to a 2-dimensional subspace $HW \subset H^{n,n}(A, \mathbb{Q})$. The rational (n, n)-classes in HW are called *Hodge-Weil classes*. The Hodge conjecture predicts that HW is spanned by classes of algebraic cycles. A *polarized abelian variety of Weil type* is a triple (A, η, h), where (A, η) is an abelian variety of Weil type and $h \in H^{1,1}(A, \mathbb{Q})$ is an ample class, such that $\eta(k) \in \mathrm{End}_{\mathbb{Q}}(A)$ maps h to $Nm(k)h$, where $Nm : K \rightarrow \mathbb{Q}$ is the norm map [vG1, W].

Polarized abelian varieties of Weil type admit a natural K-valued hermitian form $H : H_1(A, \mathbb{Q}) \times H_1(A, \mathbb{Q}) \rightarrow K$. The determinant of the matrix of H, with respect to some K-basis of $H_1(A, \mathbb{Q})$, is an element of $\mathbb{Q}^\times$ and its image in $\mathbb{Q}^\times/Nm(K^\times)$ is called the *discriminant* $\det H$ of (A, η, h). The moduli space of $2n$-dimensional abelian varieties of Weil type is n^2-dimensional. The generic abelian variety of Weil type has a cyclic Neron-Severi group, but a 3-dimensional $H^{n,n}(A, \mathbb{Q})$ [W] (see also [vG1, Th. 6.12]). The triple $(n, K, \det H)$, consisting of half the dimension of A, an imaginary quadratic number field K, and the discriminant $\det H$, is a discrete invariant of a component of the moduli space, which determines it up to isogenies of abelian varieties of Weil type [vG1, Th. 5.2(3)].

The algebraicity of the Hodge-Weil classes was proved for abelian fourfolds of Weil type with $K = \mathbb{Q}[\sqrt{-3}]$ and arbitrary discriminant in [S2], where it is also proved for abelian sixfolds with $K = \mathbb{Q}[\sqrt{-3}]$ and trivial discriminant. The algebraicity is proved for fourfolds with $K = \mathbb{Q}[\sqrt{-1}]$ and discriminant -1 in [S1] (see also [vG1] for another proof). It was later proved for sixfolds with $K = \mathbb{Q}[\sqrt{-1}]$ and discriminant -1 in [K], which implies the result for fourfolds, $K = \mathbb{Q}[\sqrt{-1}]$, and arbitrary discriminant. The algebraicity is proved in [M2] for fourfolds, arbitrary imaginary quadratic number field K, and discriminant 1.

1.2. **Abelian varieties of Weil-type from K-secants.** Let X be a projective abelian n-fold and set $\hat{X} := \mathrm{Pic}^0(X)$. Set

$$V := H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z}). \tag{1.2.1}$$

Then V is endowed with the symmetric non-degenerate unimodular bilinear pairing

$$((w_1, \theta_1), (w_2, \theta_2))_V := \theta_1(w_2) + \theta_2(w_1), \tag{1.2.2}$$

where we used the natural isomorphism to identify $H^1(\hat{X}, \mathbb{Z})$ with $H^1(X, \mathbb{Z})^*$. V has rank $4n$ and is isometric to the orthogonal direct sum $U^{\oplus 2n}$ of $2n$ copies of the even unimodular rank 2 lattice U of signature $(1, 1)$. Set $S := H^*(X, \mathbb{Z})$, $S^+ := \oplus_{i=0}^{2n} H^{2i}(X, \mathbb{Z})$, and $S^- := \oplus_{i=0}^{2n-1} H^{2i+1}(X, \mathbb{Z})$. Then S^+ and S^- are the half-spin representations of the integral spin group $\mathrm{Spin}(V)$. The spin representation S is endowed with a non-degenerate integral bilinear pairing, which is symmetric for even n and anti-symmetric for odd n

$$(s,t)_S := \int_X \tau(s) \cup t, \tag{1.2.3}$$

where the *main anti-automorphism* τ acts via multiplication by $(-1)^{i(i-1)/2}$ on $H^i(X,\mathbb{Z})$. The groups S^+ and S^- are both of rank 2^{2n-1}.

Let X be an abelian n-fold and K an imaginary quadratic number field. Set $S^+_K := S^+ \otimes_{\mathbb{Z}} K$. The spinor variety in $\mathbb{P}(S^+_K)$ parametrizes one of the two connected components of the grassmannian of maximal isotropic subspaces of V_K (see Section 2.2). Points of the spinor variety in $\mathbb{P}(S^+_K)$ are called *even pure spinors*. We are interested in lines in $\mathbb{P}(S^+_K)$, which are defined over $\mathbb{Q}$ and intersect the spinor variety at two distinct complex conjugate points. We will call such a line a *rational K-secant*, even though its points of intersection with the spinor variety are defined over K and not over $\mathbb{Q}$. When $n=2$, the spinor variety in $\mathbb{P}(S^+_{\mathbb{C}})$ is the quadric hypersurface associated to the pairing (1.2.3), and infinitely many rational secants pass through every point in $\mathbb{P}(S^+_{\mathbb{Q}})$. The rational secant line in $\mathbb{P}(\oplus_{p=0}^2 H^{p,p}(X,\mathbb{Q}))$ yielding the structure of a polarized abelian variety of Weil type on $X \times \hat{X}$ is one corresponding to a negative definite plane P in $\oplus_{p=0}^2 H^{p,p}(X,\mathbb{Q})$ with respect to the pairing (1.2.3) (see [M2]). When $n=3$, the secant variety of the spinor variety is birational to $\mathbb{P}(S^+_{\mathbb{C}})$. Through a general point λ of $\mathbb{P}(\oplus_{p=0}^3 H^{p,p}(X,\mathbb{Q}))$ passes a unique rational secant line and the latter yields the structure of an abelian variety of Weil type on $X \times \hat{X}$ only if the $\mathrm{Spin}(V_{\mathbb{Q}})$-invariant Igusa quartic, given in (10.1.1), has positive value at λ (see Section 10.1, the value of the Igusa invariant determines the field of definition of the two pure spinors). For $n \geq 4$ the image in $\mathbb{P}(S^+_{\mathbb{C}})$ of the secant variety to the spinor variety is a proper subvariety.

Following is a construction of a τ-invariant rational K-secant line to the spinor variety in all dimensions. Let Θ be an ample class on an abelian n-fold X, $n \geq 2$, and let d be a positive integer. Set $u := \sqrt{-d}\Theta$ and $K := \mathbb{Q}(\sqrt{-d})$. The automorphism of $H^*(X,K)$ of cup product with $\exp(u)$ is an element g of $\mathrm{Spin}(V_K)$. Let $\rho : \mathrm{Spin}(V_K) \to SO(V_K)$ be the standard representation. Let $P \subset \oplus_{p=0}^n H^{p,p}(X,\mathbb{Q})$ be the rational plane

$$P = \mathrm{span}_{\mathbb{Q}} \left\{ \exp(u) + \overline{\exp(u)}, \frac{\exp(u) - \overline{\exp(u)}}{\sqrt{-d}} \right\} \tag{1.2.4}$$

corresponding to the conjugation invariant plane $P_K := \mathrm{span}_K\{\exp(u), \overline{\exp(u)}\}$. The secant line $\mathbb{P}(P)$ intersects the spinor variety at $\tilde{\ell}_1 := \mathrm{span}_K\{\exp(u)\}$ and its complex conjugate $\tilde{\ell}_2 := \mathrm{span}_K\{\overline{\exp(u)}\}$. The corresponding maximal isotropic subspaces of V are $W_1 := \rho_g(H^1(\hat{X},K))$ and its complex conjugate W_2, which are calculated explicitly in (2.4.6). P yields the structure of a polarized abelian variety of Weil type on $X \times \hat{X}$ with imaginary quadratic number field K, by Proposition 2.4.4. The rational endomorphism associated to $k \in K$ via the complex multiplication

$$\eta : K \to \mathrm{End}_{\mathbb{Q}}(X \times \hat{X}) \cong \mathrm{End}(V_{\mathbb{Q}})$$

acts on W_1 by scalar multiplication by k and on W_2 by its complex conjugate $\bar{k}$.

The fact that P is spanned by Hodge classes implies that each of the intersections $W_i \cap V^{1,0}$ and $W_i \cap V^{0,1}$ is n-dimensional, for $i = 1,2$, by Lemma 2.2.6. Consequently,

$\wedge^{2n} W_i$, $i = 1, 2$, is a pair of complex conjugate lines in $H^{n,n}(X, K)$, which span the rational 2-dimensional subspace $\hat{HW}_P$ of *Hodge-Weil classes* in $H^{n,n}(X \times \hat{X}, \mathbb{Q})$.

Chevalley constructed an integral isomorphism $\tilde{\varphi} : H^*(X \times X, \mathbb{Z}) \to H^*(X \times \hat{X}, \mathbb{Z})$, given in (2.3.2), which depends on a choice of a class in $H^2(X \times \hat{X}, \mathbb{Z})$ (Remark 2.3.1). Chevalley's construction depends on the interpretation of $H^*(X \times X, \mathbb{Z})$ as the tensor square $S \otimes S$ and of $H^*(X \times \hat{X}, \mathbb{Z})$ as the exterior algebra $\wedge^* V$ and is purely representation theoretic. Consider the composition

$$\phi := (\phi_{\mathcal{P}} \otimes \phi_{\mathcal{P}}^{-1}) \circ \tilde{\varphi} \circ (id \otimes \tau) : H^*(X \times X, \mathbb{Z}) \to H^*(X \times \hat{X}, \mathbb{Z}), \tag{1.2.5}$$

where τ is the main anti-automorphism appearing in (1.2.3), $\phi_{\mathcal{P}} : H^*(X, \mathbb{Z}) \to H^*(\hat{X}, \mathbb{Z})$ is the correspondence induced by the Chern character $ch(\mathcal{P})$ of the Poincaré line bundle $\mathcal{P}$, and the left isomorphism is $\phi_{\mathcal{P}} \otimes \phi_{\mathcal{P}}^{-1} : H^*(X \times \hat{X}, \mathbb{Z}) \to H^*(\hat{X} \times X, \mathbb{Z}) \cong H^*(X \times \hat{X}, \mathbb{Z})$. The subtle Spin$(V)$-equivariance properties of ϕ will be explained in the next subsection.

Proposition 1.2.1 (Prop. 6.4.1)**.** *The image $\phi(\tilde{\ell}_i \otimes \tau(\tilde{\ell}_i))$ via $\phi \circ (id \otimes \tau)$, of the tensor square $\tilde{\ell}_i \otimes \tilde{\ell}_i$ of each of the two pure spinor lines in P, is contained in the subspace $\oplus_{k=2n}^{4n} H^k(X \times \hat{X}, K)$ and its projection to $H^{2n}(X \times \hat{X}, K)$ is $\wedge^{2n} W_i$.*

In particular, the isomorphism $\phi \circ (id \otimes \tau)$ maps the 2-dimensional rational subspace $HW_P := [\tilde{\ell}_1 \otimes \tilde{\ell}_1] \oplus [\tilde{\ell}_2 \otimes \tilde{\ell}_2]$ of $P \otimes P \subset H^{even}(X \times X, \mathbb{Q})$ to a 2-dimensional subspace of $H^{even}(X \times \hat{X}, \mathbb{Q})$, and the latter projects onto the 2-dimensional subspace $\hat{HW}_P$ of Hodge-Weil classes in $H^{2n}(X \times \hat{X}, \mathbb{Q})$.

1.3. **Orlov's equivalence $\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$.** Let $\mathcal{P}$ be the Poincaré line bundle over $X \times \hat{X}$. Let $\mu : X \times X \to X \times X$ be given by $\mu(x, y) = (x + y, y)$. Let $id \times \Phi_{\mathcal{P}} : D^b(X \times \hat{X}) \to D^b(X \times X)$ be the equivalence with Fourier-Mukai kernel $\mathcal{O}_{\Delta_X} \boxtimes \mathcal{P}$, where Δ_X is the diagonal in $X \times X$. *Orlov's equivalence*

$$\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$$

is the inverse of

$$\mu_* \circ (id \times \Phi_{\mathcal{P}}) : D^b(X \times \hat{X}) \to D^b(X \times X).$$

Let $m : \mathrm{Spin}(V) \to GL[H^*(X, \mathbb{Z})]$ be the spin representation and define $m^\dagger : Spin(V) \to GL[H^*(X, \mathbb{Z})]$ by $m_g^\dagger = \tau m_g \tau$. Let $\rho : \mathrm{Spin}(V) \to SO(V)$ be the standard representation and denote the induced representation on $H^*(X \times \hat{X}, \mathbb{Z}) \cong \wedge^* V$ by ρ as well. Define the representation

$$\rho' : \mathrm{Spin}(V) \to GL[H^*(X \times \hat{X}, \mathbb{Z})]$$

by $\rho'_g = \exp\left(\frac{1}{2}[c_1(\mathcal{P}) - \rho_g(c_1(\mathcal{P}))]\right) \rho_g$, for all $g \in \mathrm{Spin}(V)$. The two integral Spin(V)-representation ρ and ρ' are non-isomorphic, but they are isomorphic once tensored with

$\mathbb{Q}$, as the upper square in the following diagram is commutative, for all $g \in \mathrm{Spin}(V)$.

$$
\begin{tikzcd}
H^*(X\times\hat{X},\mathbb{Q}) \arrow[r, "\rho_g"] & H^*(X\times\hat{X},\mathbb{Q}) \\
H^*(X\times\hat{X},\mathbb{Q}) \arrow[u, "\cup\exp\left(-\frac{1}{2}c_1(\mathcal{P})\right)"] \arrow[r, "\rho'_g"'] & H^*(X\times\hat{X},\mathbb{Q}) \arrow[u, "\cup\exp\left(-\frac{1}{2}c_1(\mathcal{P})\right)"'] \\
H^*(X\times X,\mathbb{Q}) \arrow[u, "\phi\circ(id\otimes\tau)"] \arrow[r, "m_g\otimes m_g^\dagger"'] \arrow[uu, bend left=60, "\tilde{\phi}"] & H^*(X\times X,\mathbb{Q}) \arrow[u, "\phi\circ(id\otimes\tau)"'] \arrow[uu, bend right=60, "\tilde{\phi}"']
\end{tikzcd}
\tag{1.3.1}
$$

The commutativity of the lower square and the $\mathrm{Spin}(V)$-equivariance of

$$\tilde{\phi} := \exp\left(-c_1(\mathcal{P})/2\right) \cup \phi\circ(id\otimes\tau) : H^*(X\times X,\mathbb{Q}) \to H^*(X\times\hat{X},\mathbb{Q}) \tag{1.3.2}$$

are established by the following result. In particular, $\tilde{\phi}$ is $\mathrm{Spin}(V)$-equivariant, where the domain is the representation $m\otimes m^\dagger$ and the codomain is ρ.

Proposition 1.3.1 (Proposition 6.1.2)**.** *The isomorphism*

$$\phi : H^*(X\times X,\mathbb{Z}) \to H^*(X\times\hat{X},\mathbb{Z}),$$

given in (1.2.5), is equal to the isomorphism induced by Φ *and* $\phi\circ(id\otimes\tau)$ *is* $\mathrm{Spin}(V)$*-equivariant, where the domain is the representation* $m\otimes m^\dagger$ *and the codomain is* ρ'*.*

An object F of $D^b(X)$ is called a *P-secant object*, if $ch(F)$ belongs to a secant plane P. Given two P-secant objects, we refer to the object $E := \Phi(F_1\boxtimes F_2^\vee)$ as a *P-secant$^{\boxtimes 2}$-object* in $D^b(X\times\hat{X})$. Let $\mathrm{Spin}(V)_P$ be the subgroup of $\mathrm{Spin}(V)$ leaving every element of the plane P invariant. When P is given by (1.2.4) the subspace $H^2(X\times\hat{X},\mathbb{Q})^{\mathrm{Spin}(V)_P}$ is one-dimensional spanned by an ample class h, by Lemma 2.4.4. Given an element $k\in K$, the rational endomorphism $\eta(k)\in \mathrm{End}_{\mathbb{Q}}(X\times\hat{X})$ maps h to $Nm(k)h$, where $Nm: K\to\mathbb{Q}$ is the norm map. Consequently, $(X\times\hat{X},\eta,h)$ is a polarized abelian variety of Weil type. The period domain of deformations of $(X\times\hat{X},\eta,h)$ as a polarized abelian variety of Weil type is the adjoint orbit of the complex structure of $X\times\hat{X}$ in $\mathrm{Spin}(V_{\mathbb{R}})_P$, by Lemma 4.0.2 and Corollary 4.0.4.

Given a class ch in $H^{ev}(X\times\hat{X},\mathbb{Q})$ with graded summand ch_i in $H^{2i}(X\times\hat{X},\mathbb{Q})$ and with $ch_0 = r\neq 0$ considered as a rational number, set $\kappa(ch) = \exp(-ch_1/r)ch$. Given an object in $D^b(X\times\hat{X})$ of non-zero rank set $\kappa(E) := \kappa(ch(E))$. The following observation motivated the current paper. It is an immediate corollary of the above proposition.

Corollary 1.3.2. *If the rank r of a P-*secant$^{\boxtimes 2}$-object $E := \Phi(F_1\boxtimes F_2^\vee)$ *is non zero, then its characteristic class $\kappa(E)$ is $\mathrm{Spin}(V)_P$-invariant with respect to the representation ρ. Consequently, $\kappa(E)$ remains of Hodge-type, under every deformations of $(X\times\hat{X},\eta,h)$ as a polarized abelian variety of Weil-type.*

Proof. Let g be an element of $\mathrm{Spin}(V)_P$. Using the fact that ρ_g is an algebra automorphism we have $\rho_g(\exp(\bullet)) = \exp(\rho_g(\bullet))$ and $\rho_g(\kappa(\bullet)) = \kappa(\rho_g(\bullet))$. The fact that F_1 and F_2 are P-secant sheaves implies that $m_g\otimes m_g^\dagger$ leaves $ch(F_1\otimes F_2^\vee)$ invariant. Hence $\rho'_g(ch(E)) = ch(E)$, by Proposition 6.1.2. Thus $ch(E)\cup\exp\left(-\frac{1}{2}c_1(\mathcal{P})\right)$ is ρ_g

invariant, $ch(E) \cup \exp\left(-\frac{1}{2}c_1(\mathcal{P})\right) = \rho_g(ch(E)) \cup \exp\left(-\frac{1}{2}\rho_g(c_1(\mathcal{P}))\right)$, by the commutativity of the upper square in Diagram (1.3.1). Applying κ to both sides, we get $\kappa(ch(E)) = \kappa(\rho_g(ch(E))) = \rho_g(\kappa(ch(E)))$. Hence, $\kappa(E) = \rho_g(\kappa(E))$.

Finally, $\mathrm{Spin}(V)_P$-invariant classes remain of Hodge-type under every deformation of $(X \times \hat{X}, \eta, h)$ as a polarized abelian variety of Weil-type, by Corollary 4.0.4. □

1.4. **Ideal secant sheaves on the Jacobian of a genus 3 curve.** We consider the example where X is the Jacobian $\mathrm{Pic}^2(C)$ of a genus 3 non-hyperelliptic curve C, $F_1 = \mathcal{I}_{\cup_{i=1}^{d+1} C_i}(\Theta)$ is the tensor product of the theta line bundle with the ideal sheaf of the union of $d+1$ generic translates $C_i \subset X$ of the Abel Jacobi image $AJ(C)$ of C, for $d \geq 3$, and $F_2 = \mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}(\Theta)$, where $\Sigma_i \subset X$ is a generic translate of $-AJ(C)$. Set $u := \sqrt{-d}\Theta$, $d \geq 3$. We prove the following:

Theorem 1.4.1. (1) *The line $\mathbb{P}$ in $\mathbb{P}H^{even}(X, \mathbb{Q})$ through $ch(F_1^\vee)$ and $ch(F_2)$ intersects the spinor variety at the two complex conjugate pure spinors* $\exp(u)$ *and* $\overline{\exp(u)}$ *defined over $K = \mathbb{Q}(\sqrt{-d})$ (Lemma 8.1.1).*

(2) *The dual object $\Phi(F_2 \boxtimes F_1)^\vee$ is isomorphic to $\mathcal{E}[-2]$, where $\mathcal{E}$ is a simple reflexive sheaf of rank $8d$ over $X \times \hat{X}$ (Proposition 9.2.2).*

(3) *The characteristic class $\kappa(\mathcal{E}) := \exp(-c_1(\mathcal{E})/\mathrm{rank}(\mathcal{E}))ch(\mathcal{E})$ remains of Hodge type under every deformation of $(X \times \hat{X}, \eta, h)$ as a polarized abelian sixfold of Weil type (Lemma 6.2.3).*

(4) *The $\eta(K)$-translates of the graded summand $\kappa_3(\mathcal{E})$ of $\kappa(\mathcal{E})$ in $H^{3,3}(X \times \hat{X}, \mathbb{Q})$, together with h^3, span the 3-dimensional subspace $\mathbb{Q}h^3 \oplus H\hat{W}_P$ (Proposition 6.4.1 and Lemma 8.2.1).*

(5) *$\mathcal{E}$ deforms with $(X \times \hat{X}, \eta, h)$ to first order as a twisted sheaf in every direction in the 9-dimensional moduli space of polarized abelian sixfolds of Weil type (Lemma 8.3.1 (2) and Remark 8.4.4).*

Orlov's derived equivalence $\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$ relates diagonal generalized (including non-commutative and gerby) deformations of $D^b(X \times X)$ along which the objects F_i, $i = 1, 2$, deform, to *commutative* deformations of $X \times \hat{X}$ as polarized abelian varieties of Weil type (with an additional gerby structure), by Corollary 8.4.2. The proof of (5) above reduces to showing that F_i deforms to first order along a 9-dimensional subspace of $HH^2(X)$. This amounts to showing that the obstruction map $ob_{F_i} : HH^2(X) \to \mathrm{Ext}^2(F_i, F_i)$ has rank 6 (Proposition 8.2.9).

1.5. **Semiregular K-secant sheaves.** Let E be a coherent sheaf on a N-dimensional compact Kähler manifold Y. Denote the Atiyah class of E by $at_E \in \mathrm{Ext}^1(E, E \otimes \Omega^1_Y)$. The q-th component σ_q of the *semiregularity map*

$$\sigma := (\sigma_0, \ldots, \sigma_{N-2}) : \mathrm{Ext}^2(E, E) \to \prod_{q=0}^{N-2} H^{q+2}(Y, \Omega^q_Y)$$

is the composition $\mathrm{Ext}^2(E, E) \xrightarrow{(at_E)^q/q!} \mathrm{Ext}^{q+2}(E, E \otimes \Omega^q_Y) \xrightarrow{Tr} H^{q+2}(Y, \Omega^q_Y)$. The sheaf E is said to be *semiregular*, if σ is injective. Buchweitz and Flenner proved that when

E is semiregular the pair (Y, E) deforms locally in the sublocus in the smooth base of a deformation of Y, where $ch(E)$ remains of Hodge-type [BF1, Th. 5.1].

If E is locally free, then the projective bundle $\mathbb{P}(E)$ often deforms over a larger family relaxing the condition that $c_1(E)$ remains of Hodge-type $(1,1)$. Every $\mathbb{P}^{r-1}$-bundle admits a lift to a locally free sheaf twisted by a Čech 2-cocycle with coefficients in the local system μ_r of r-th roots of unity and with trivial determinant. The Buchweitz-Flenner theorem has a generalization for twisted sheaves [Pr, Remark 2.6]. We formulate a conjectural generalization of the Buchweitz-Flenner theorem for such twisted sheaves, which we believe[1] follows from Pridham's result. The usual definition of the Atiyah class goes through for such a sheaf (Definition 7.3.5) and the semiregularity map is defined as in the untwisted case. Conjecture 7.3.9 states that E deforms over the sublocus of the base of the deformation, where $\kappa(E)$ remains of Hodge type. Note that if E is locally free, then $\kappa(E)$ is the trace of the exponential Atiyah class of $\mathbb{P}(E)$, in precise analogy with the relationship between $ch(E)$ and the Atiyah class of E for an untwisted sheaf E. We prove Conjecture 7.3.9 for twisted sheaves on abelian varieties in Section 7.4.

Keep the notation of Theorem 1.4.1. The sheaf $\mathcal{E}$ in the theorem is not semiregular. We describe next the equivariant version of the construction, carried out in Section 9.3, which produces a semiregular sheaf. Let G_1 and G_2 be cyclic subgroups of X of order $d+1$ with $G_1 \cap G_2 = (0)$. Choose $C_i \subset X$, $1 \leq i \leq d+1$, to be a G_1-orbit of translates of the Abel-Jacobi image $AJ(C)$ of C. Choose $\Sigma_i \subset X$, $1 \leq i \leq d+1$, to be a G_2-orbit of translates of $-AJ(C)$. The ideal sheaf $\mathcal{I}_{\cup_{i=1}^{d+1} C_i}$ is G_1-equivariant, $\mathcal{I}_{\cup_{i=1}^{d+1} \Sigma_i}$ is G_2-equivariant, and $\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{d+1} \Sigma_i}$ is $G_1 \times G_2$-equivariant. The reflexive sheaf $\mathcal{E}$ is the dual of the image of $\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{d+1} \Sigma_i}$ via an equivalence $\tilde{\Phi}$ (the composition of Φ with tensorization by $\Theta \boxtimes \Theta$ and a shift) between $D^b(X \times X)$ and $D^b(X \times \hat{X})$ and $\mathcal{E}^\vee$ is thus equivariant with respect to the image $G \subset (X \times \hat{X}) \times \mathrm{Pic}^0(X \times \hat{X})$ of $G_1 \times G_2$ via the Rouqier isomorphism associated to $\tilde{\Phi}$ between the identity components of the groups of autoequivalences of $X \times X$ and $X \times \hat{X}$. Denote by $\bar{G}$ the image of G via the projection to the cartesian factor $X \times \hat{X}$. The projection $G \to \bar{G}$ is an isomorphism, by Lemma 9.3.3.

Let $q : X \times \hat{X} \to Y := (X \times \hat{X})/\bar{G}$ be the quotient morphism. When d is even[2] the rank $8d$ of $\mathcal{E}$ is relatively prime to the order $(d+1)^2$ of $\bar{G}$. In that case we can replace $\mathcal{E}$ by its tensor product with a suitable power of the line bundle $\det(\mathcal{E})$ to get a $\bar{G}$-equivariant sheaf $\tilde{\mathcal{E}}$. The latter descends to a semiregular reflexive sheaf $\bar{\mathcal{E}}$ over Y satisfying $q^*\bar{\mathcal{E}} = \tilde{\mathcal{E}}$. Finally, we associate to $\bar{\mathcal{E}}$ a reflexive sheaf $\mathcal{B}$, twisted by a Čech 2-cocycle with coefficients in μ_{8d} and with a trivial determinant line bundle, satisfying $\kappa(\mathcal{E}) = q^*\kappa(\mathcal{B})$. Such a sheaf $\mathcal{B}$ is constructed, regardless of the parity of d. The morphism q induces a local isomorphism of the Kuranishi deformation spaces of $X \times \hat{X}$ and Y. Our verification of Conjecture 7.3.9 in the case of abelian varieties implies that $(Y, \mathcal{B})$ deforms locally over the locus where $\kappa(\mathcal{B})$ remains of Hodge-type. Hence, $(X \times \hat{X}, q^*\mathcal{B})$ deforms locally over the locus where $q^*\kappa(\mathcal{B})$ remains of Hodge type. Now $\kappa(\mathcal{E}) = q^*\kappa(\mathcal{B})$ and $\kappa(\mathcal{E})$ remains of Hodge type over the locus, where $(X \times \hat{X}, \eta, h)$

[1] One needs to verify that our semiregularity map is injective, if and only if Pridham's is.

[2] If d is odd replace it with $4d$ and note that $\mathbb{Q}(\sqrt{-4d}) = \mathbb{Q}(\sqrt{-d})$.

deforms as an abelian variety of Weil-type, by Corollary 1.3.2, yielding a proof of the the following.

Theorem 1.5.1. *Let d be a positive integer. Set $K := \mathbb{Q}(\sqrt{-d})$. The Hodge-Weil classes of polarized abelian sixfolds of Weil type with complex multiplication by K and with discriminant -1 are algebraic.*

The Theorem is proved in Section 9.3. The construction described above is generalized in Example 8.1.3 to produce secant$^{\boxtimes 2}$-objects over $X \times \hat{X}$, for Jacobians X of higher genus curves, but the proof of their semiregularity is special to genus 3.

1.6. **Verification of the Hodge conjecture for abelian fourfolds.** The following is a known consequence of Theorem 1.5.1.

Corollary 1.6.1. *The Hodge conjecture holds for abelian fourfolds.*

Proof. Theorem 1.5.1 implies that the Hodge-Weil classes are algebraic for every abelian fourfold of Weil-type, for all imaginary quadratic number fields, and for all discriminants, by degenerating abelian sixfolds of Weil type of discriminant -1 to products of abelian fourfolds of Weil type of arbitrary discriminant and abelian surfaces of Weil type [S2, Prop. 10]. If A is a simple abelian fourfold, then $H^{2,2}(A, \mathbb{Q})$ is spanned by quadratic polynomials in divisor classes and by Hodge-Weil classes (for possibly infinitely many complex multiplications), by [MZ1, Theorem 2.11]. The Hodge conjecture for abelian fourfolds is thus reduced to the case of non-simple abelian fourfolds. If the Hodge conjecture is verified for an abelian variety, then it follows for any isogenous abelian variety. The Hodge conjecture for the product of two abelian surfaces was proved in [R, Theorem 4.11]. If $A = B \times E$, where B is a simple abelian 3-fold and E is an elliptic curve, then either the Hodge ring is generated by divisor classes, or E and B both have complex multiplication by the same imaginary quadratic number field K, by [MZ3, Prop. 3.8], and in the latter case the Hodge ring of A is generated by divisor classes and Hodge-Weil classes, by [MZ3, Theorem 0.1(i)]. □

1.7. **Organization of the paper.** In Sections 2 to 4 we reformulate Weil's construction of abelian varieties of Weil type, their special Mumford-Tate group, and their period domain, in terms of $\mathrm{Spin}(V)$ representations and a rational K-secant line P. In Section 2 we associate to a non-degenerate rational K-secant $P \subset H^*(X, \mathbb{Q})$, which is spanned by Hodge classes, the structure of a polarized abelian variety of Weil type on $X \times \hat{X}$ and show that $\mathrm{Spin}(V)_P$ plays the role of the special Mumford-Tate group of a generic abelian variety of Weil type in its deformation class. We denote the $\mathrm{Spin}(V)_P$-invariant polarization by Ξ and the polarized abelian variety of Weil type by $(X \times \hat{X}, \Xi, \eta)$.

In Section 3 we construct a $\mathrm{Spin}(V)_P$-invariant K-valued hermitian form H on the vector space $H^1(X \times \hat{X}, \mathbb{Q})$. In section 4 we show that the period domain of deformations of $(X \times \hat{X}, \Xi, \eta)$ as a polarized abelian variety of Weil type is the adjoint orbit of the complex structure I of $X \times \hat{X}$ in $\mathrm{Spin}(V_{\mathbb{R}})_P$.

In Section 5 we review general results about the group of autoequivalences of the derived category of an abelian variety.

Section 6 is dedicated to the study of the isomorphism $\phi : H^*(X\times X,\mathbb{Z}) \to H^*(X\times\hat{X})$ induced by of Orlov's equivalence $\Phi : D^b(X\times X) \to D^b(X\times\hat{X})$. We describe the $\mathrm{Spin}(V)$-equivariance properties of ϕ, relate it to Chevalley's isomorphism $S\otimes S \to \wedge^*V$, and use it to conclude that ϕ maps the 2-dimensional subspace of $H^*(X\times X, K)$, spanned by tensor squares of pure spinors in $P\otimes_{\mathbb{Q}} K$, onto the 2-dimensional space of Hodge-Weil classes.

In Section 7 we review the theorem of Buchweitz-Flenner verifying the variational Hodge conjecture for semiregular sheaves. We then formulate a conjectural generalization for semiregular sheaves of rank $r > 0$ twisted by a Čech 2-cocycle with coefficients in the local system μ_r of r-th roots of unity (Conjecture[3] 7.3.9). We prove the conjecture for families of abelian varieties by reducing it to the original result of Buchweitz-Flenner via an elementary argument.

In Section 8 we concentrate on the case where X is the Jacobian of a non-hyperelliptic curve C of genus 3. We construct a $\mathbb{Q}(\sqrt{-d})$-secant sheaf F on X, for every positive integer d, as the sheaf $\mathcal{I}_{\cup_{i=1}^{d+1} C_i}(\Theta)$, where Θ is the canonical principal polarization and C_i, $1 \leq i \leq d+1$, are disjoint translates of the Abel-Jacobi image of C in X. Recall that both the Yoneda algebra $\mathrm{Ext}^*(F,F)$ and the cohomology $H^*(X,\mathbb{C})$ are modules over the Hochschild cohomology $HH^*(X)$. The space $HH^2(X)$ parametrized generalized deformations of the abelian category of coherent sheaves on X. We show that the kernel of the evaluation homomorphism $ob_F : HH^2(X) \to \mathrm{Ext}^2(F,F)$ is equal to the kernel of the evaluation homomorphism $ch(F)\rfloor : HH^2(X) \to H^*(X,\mathbb{C})$. In particular, F deforms to first order in every direction in which $ch(F)$ remains of Hodge-type. We then extend the equality $\ker(ob_F) = \ker(ch_F)$, replacing X by $X\times\hat{X}$ and F by the secant$^{\boxtimes 2}$-object $\mathcal{G} := \Phi(F_2 \boxtimes F_1)[3]$ over $X\times\hat{X}$, to obtain $\ker(ob_{\mathcal{G}}) = \ker(ch(\mathcal{G}))$, where F_i are the secant sheaves in Theorem 1.4.1 (see Lemma 8.3.1). We conclude that $\mathcal{G}$ deforms to first order in all directions in $H^1(X\times\hat{X}, T[X\times\hat{X}])$ tangent to the period domain of deformations of $(X\times\hat{X},\Xi,\eta)$ as a polarized abelian variety of Weil type.

In Section 9 we complete the proof of Theorem 1.4.1 and prove Theorem 1.5.1. We show that $\mathcal{E} := \mathcal{G}^\vee[-1]$ is a reflexive sheaf, under a general position assumption on the curves C_i and Σ_j in Theorem 1.4.1. We also check that the general position assumption can be achieved with a G_1-equivariant $\mathcal{I}_{\cup_{i=1}^{d+1} C_i}$ and a G_2-equivariant $\mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}$ in Theorem 1.4.1, where G_1 and G_2 are cyclic subgroups of X of order $d+1$. We show that $\mathcal{E}$ can be then normalized to a $\bar{G}$-equivariant sheaf over $X\times\hat{X}$, which descends to a semiregular simple reflexive sheaf $\mathcal{B}$ over $(X\times\hat{X})/\bar{G}$, twisted by a Čech 2-cocycle with coefficients in μ_{8d}, where $\bar{G}$ is a subgroup of $X\times\hat{X}$ isomorphic to $G_1\times G_2$. The semiregularity of $\mathcal{B}$ follows from the equality $\ker(ob_{\mathcal{G}}) = \ker(ch(\mathcal{G}))$ mentioned above and the surjectivity of $ob_{\mathcal{G}} : HH^2(X\times\hat{X}) \to \mathrm{Ext}^2(\mathcal{G},\mathcal{G})^{G_1\times G_2}$ (Lemma 9.3.11). Finally we derive from the semiregularity theorem the algebraicity of the Hodge-Weil classes for polarized abelian sixfolds of Weil-type with discriminant -1 and with complex multiplication by an arbitrary imaginary quadratic number field (Theorem 1.5.1).

In Section 10 we characterize K-secant sheaves on abelian 3-folds, by the value of the Igusa $\mathrm{Spin}(V)$-invariant polynomial on their Chern character. In the appendix

[3] As mentioned, we expect the conjecture to follow from [Pr, Remark 2.26].

section 11 we prove that the derived tensor product of the sheaves $\mathcal{I}_{\cup_{i=1}^{d+1}C_i}$ and $\mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}$ in Theorem 1.5.1 is isomorphic to the ideal sheaf of a 1-dimensional subscheme, even when the subschemes $\cup_{i=1}^{d+1}C_i$ and $\cup_{i=1}^{d+1}\Sigma_i$ intersect.

2. Abelian $2n$-folds of Weil type associated to a rational secant to the even spinor variety of an abelian n-fold

In Section 2.1 we recall the definition of the Clifford algebra $C(V)$ and of the Spin group as a subgroup of the group of invertible elements of $C(V)$. We recall also that when $V = H^1(X \times \hat{X}, \mathbb{Z})$, then $S := H^*(X, \mathbb{Z})$ is the Spin representation and $S^+ := H^{ev}(X, \mathbb{Z})$ and $S^- := H^{odd}(X, \mathbb{Z})$ are its two half-spin subrepresentations. In Section 2.2 we recall the notion of pure spinors, which are elements of the half-spin representations corresponding to maximal isotropic subspaces of $V_{\mathbb{C}}$. The set of even pure spinors in $\mathbb{P}(S^+_{\mathbb{C}})$ is the spinorial variety. We show that a rational secant line P to the spinorial variety, which intersects it at non-rational points, determines a K-vector space structure on $V_{\mathbb{Q}}$, for a quadratic field extension K of $\mathbb{Q}$. We determine also the subalgebra of $\wedge^* V$ invariant under the subgroup $\mathrm{Spin}(V)_P$ leaving all vectors in the 2-dimensional subspace $P \subset S^+_{\mathbb{Q}}$ invariant. In Section 2.3 we recall the isomorphism $S \otimes_{\mathbb{Z}} S \cong \wedge^* V$ of $\mathrm{Spin}(V)$-representations. In Section 2.4 we give, for every imaginary quadratic number field K, an example of a secant to the spinorial variety, such that the associated K-vector space structure on $V_{\mathbb{Q}} = H^1(X \times \hat{X}, \mathbb{Q})$ makes $X \times \hat{X}$ a polarized abelian variety of Weil type.

2.1. **The Clifford algebra and the spin group.** Keep the notation of Section 1.2. Let V be the lattice given in (1.2.1). Let $C(V)$ be the Clifford algebra, the quotient of the tensor algebra $\oplus_{i=0}^{\infty} V^{\otimes i}$ by the relation

$$v_1 \cdot v_2 + v_2 \cdot v_1 = (v_1, v_2)_V, \tag{2.1.1}$$

where the integer in the right hand side is regarded in $V^{\otimes 0} \cong \mathbb{Z}$. Then $C(V)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra with a unit.

Given $w \in H^1(X, \mathbb{Z})$, define the endomorphism $L_w : S \to S$ of degree 1 by $L_w(\bullet) = w \wedge (\bullet)$. Given $\theta \in H^1(X, \mathbb{Z})^*$ we get the endomorphism $D_\theta : S \to S$ of degree -1 given by contraction with θ. We get the embedding

$$m : V \to \mathrm{End}(S), \tag{2.1.2}$$

given by $m_{(w,\theta)} := L_w + D_\theta$ and satisfying the analogue

$$m_{v_1} \circ m_{v_2} + m_{v_2} \circ m_{v_1} = (v_1, v_2)_V \cdot id_S$$

of the Clifford relation (2.1.1). Hence, m extends an algebra homomorphism

$$m : C(V) \to \mathrm{End}(S), \tag{2.1.3}$$

which is in fact an isomorphism, by [GLO, Prop. 3.2.1(e)]. The *main anti-automorphism*

$$\tau : C(V) \to C(V)$$

sends $v_1 \cdot \dots \cdot v_r$ to $v_r \cdot \dots \cdot v_1$. The main involution $\alpha : C(V) \to C(V)$ acts by multiplication by -1 on $C(V)^{odd}$ and as the identity on $C(V)^{even}$. The *conjugation*

$x \mapsto x^*$ is the composition of τ and α. Let $C(V)^\times$ be the group of invertible elements in $C(V)$. The integral Clifford group is

$$G(V) := \{x \in C(V)^\times \ : \ x \cdot V \cdot x^{-1} \subset V\}.$$

The integral spin group is its index four subgroup

$$\mathrm{Spin}(V) := \{x \in C(V)^{even} \ : \ x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}.$$

The standard representation $\rho : G(V) \to O(V)$ of $G(V)$ is defined by $\rho(x)(v) = x \cdot v \cdot x^{-1}$. If $(v,v) = \pm 2$, then $-\rho(v)$ is the reflection with respect to the co-rank one lattice $v^\perp$ orthogonal to v,

$$-\rho(v)(\lambda) = \lambda - \frac{2(\lambda, v)_V}{(v,v)_V} \cdot v, \ \forall \lambda \in V.$$

If $(v_1, v_1)_V = (v_2, v_2)_V = 2$, or $(v_1, v_1)_V = (v_2, v_2)_V = -2$, then $v_1 \cdot v_2$ belongs to $\mathrm{Spin}(V)$ and the group $\mathrm{Spin}(V)$ is generated by such elements.

An element $x \in C(V)^{odd}$ is mapped via (2.1.3) to an endomorphism m_x of S, which maps S^+ to S^- and S^- to S^+. In particular, we get the $\mathrm{Spin}(V)$ equivariant homomorphisms $V \otimes S^+ \to S^-$ and $V \otimes S^- \to S^+$. Given an element $v \in V$, denote by $m_{v,+-} : S^+ \to S^-$ and by $m_{v,-+} : S^- \to S^+$ the resulting homomorphisms. The latter are adjoints with respect to the bilinear pairing (1.2.3). More generally, for $s, t \in S$ and $v \in V$ we have

$$(m_v(s), t)_S = (s, m_v(t))_S,$$

by [Ch, III.2.2]. Given an element $w \in S^+$, denote by $m_w : V \to S^-$ the homomorphism given by

$$m_w(v) := m_{v,+-}(w).$$

Given $w \in S^-$, define $m_w : V \to S^+$ by $m_w(v) = m_{v,-+}(w)$. The norm character $N : G(V) \to \{\pm 1\}$ is given by $N(g) = g \cdot \tau(g)$. For $g \in G(V)$ and $s, t \in S$ we have

$$(g(s), g(t))_S = N(g)(s,t)_S,$$

by [Ch, III.2.1]. In particular, if $v \in V$ and $(v,v)_V = 2$, then v belongs to $G(V)$, $N(v) = 1$, $m_v : S \to S$ is an isometry interchanging S^+ and S^-, and $m_v^2 = \mathbb{1}_S$.

Given a field K we set $V_K := V \otimes_{\mathbb{Z}} K$, and define S_K, S_K^+, and S_K^- similarly. The same definitions above yield the Clifford algebra $C(V_K)$, the groups $G(V_K)$, $\mathrm{Spin}(V_K)$, and $O(V_K)$, the representation $\rho : G(V_K) \to O(V_K)$, and the isomorphism $m : C(V_K) \to \mathrm{End}(S_K)$.

2.2. **Rational lines K-secant to the variety of even pure spinors.** An element $w \in S^+_{\mathbb{C}}$ is called an *even pure spinor*, if the kernel of $m_w : V_{\mathbb{C}} \to S^-_{\mathbb{C}}$ is a maximal isotropic subspace. An element $w \in S^-_{\mathbb{C}}$ is called an *odd pure spinor*, if the kernel of $m_w : V_{\mathbb{C}} \to S^+_{\mathbb{C}}$ is a maximal isotropic subspace [Ch, III.1.4]. Every maximal isotropic subspace is of this form and so the $(2n^2 - n)$-dimensional grassmannian $IGr(2n, V_{\mathbb{C}})$ of $2n$-dimensional isotropic subspaces has two connected components $IGr_+(2n, V_{\mathbb{C}})$ and $IGr_-(2n, V_{\mathbb{C}})$ [Ch, III.1.5]. We get a $Spin(V)$-equivariant embedding

$$IGr_+(2n, V_{\mathbb{C}}) \ \to \mathbb{P}(S^+_{\mathbb{C}}) \cong \mathbb{P}^{2^{2n-1}-1}$$

sending $W \in IGr_+(2n, V_{\mathbb{C}})$ to the line spanned by an even pure spinor s with $W = \ker(m_s)$. The image of the embedding in $\mathbb{P}(S^+_{\mathbb{C}})$ is called the (even) *spinor variety*.

Let K be a purely imaginary quadratic number field. Let ℓ_1, ℓ_2 be two complex conjugate points in $\mathbb{P}(S^+_K)$. Assume that the line $\tilde{\ell}_i$ in S^+_K corresponding to ℓ_i is spanned by a pure spinor. Let $W_i \subset V_K$ be the maximal isotropic subspace of V_K corresponding to ℓ_i. Note that W_2 is the complex conjugate of W_1. Let P_K be the plane in S^+_K spanned by $\tilde{\ell}_1$ and $\tilde{\ell}_2$. P_K is defined over $\mathbb{Q}$ and we denote by P the corresponding subspace of $S^+_{\mathbb{Q}}$.

Lemma 2.2.1. *P is isotropic with respect to the pairing $(\bullet, \bullet)_S$, given in (1.2.3), if and only if $W_1 \cap W_2 \neq (0)$. The restriction of the pairing $(\bullet, \bullet)_S$ to P is definite, if and only if $W_1 \cap W_2 = (0)$.*

Proof. Let λ_1 be a non-zero element of $\tilde{\ell}_1$ and set $\lambda_2 = \bar{\lambda}_1$. Then $(\lambda_i, \lambda_i) = 0$, for $i = 1, 2$, by [Ch, III.2.4]. Furthermore, $(\lambda_1, \lambda_2) = 0$, if and only if $W_1 \cap W_2 \neq (0)$, by [Ch, III.2.4]. Write $\lambda_1 = a + ib$, with $a, b \in S^+_{\mathbb{Q}}$. Then

$$\begin{aligned} (\lambda_1, \lambda_1) &= (a, a)_S - (b, b)_S + 2i(a, b)_S = 0, \\ (\lambda_1, \lambda_2) &= (a, a)_S + (b, b)_S. \end{aligned}$$

The first equation implies that $(a, a) = (b, b)$ and $(a, b) = 0$. The second equation thus implies that $(a, a) = 0$, if and only if $W_1 \cap W_2 \neq 0$. □

Assume that $W_1 \cap W_2$ is the zero subspace. Then $\{\ell_1, \ell_2\}$ is the set theoretic intersection of $IGr_+(2n, V_{\mathbb{C}})$ and the line through ℓ_1 and ℓ_2, provided $n > 1$, by [Ch, III.1.12]. Denote by $\mathrm{Spin}(V_K)_{\ell_i}$ the subgroup of $\mathrm{Spin}(V_K)$ stabilizing ℓ_i. Denote their intersection by

$$\mathrm{Spin}(V_K)_{\ell_1,\ell_2} := \mathrm{Spin}(V_K)_{\ell_1} \cap \mathrm{Spin}(V_K)_{\ell_2}. \tag{2.2.1}$$

The quotient $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}/\{\pm 1\}$ is isomorphic to $GL(W_1)$, and so to $GL_{2n}(K)$, and the quotient maps injectively into $SO^+(V_K)$, by the proof of [I, Lemma 1]. The element of $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}/\{\pm 1\}$ corresponding to $g \in GL(W_1)$ acts on W_2 via $(g^*)^{-1}$, where W_2 is identified with W_1^* via the bilinear pairing of V_K. Let

$$\det_i : \mathrm{Spin}(V_K)_{\ell_1,\ell_2} \to K^\times$$

be the pullback from $GL(W_i)$ of the determinant character. Then $\det_2(s) = \det_1(s)^{-1}$. The characters $\tilde{\ell}_i$ and $\det_i$ satisfy $\tilde{\ell}_i \otimes \tilde{\ell}_i \cong \det_i$, by the proof of [I, Lemma 1] (see the second displayed formula for $\phi(s_i(\lambda))$ in [I, Sec. 2]. See also [Ch, III.3.2]).

Denote by $\mathrm{Spin}(V_K)_P$ the subgroup of $\mathrm{Spin}(V_K)$ leaving every vector in P_K invariant. Define

$$\mathrm{Spin}(V)_P \tag{2.2.2}$$

and $\mathrm{Spin}(V_{\mathbb{Q}})_P$ analogously.

Lemma 2.2.2. *The group $\mathrm{Spin}(V_K)_P$ is isomorphic to $SL_n(K)$.*

Proof. Note that $\mathrm{Spin}(V_K)_P$ is the kernel of $\det_1$ in $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ and so it is isomorphic to $SL_n(K)$, since -1 is not contained in $\mathrm{Spin}(V_K)_P$. □

Remark 2.2.3. ([I, Lemma 2] and the remark following it). Assume that $n \geq 3$. Let w be a point in P. Assume that w does not belong to neither $\tilde{\ell}_1$ nor $\tilde{\ell}_2$. Then w is not a pure spinor, as commented above. If n is odd, then the stabilizer of w in $\mathrm{Spin}(V_K)$ is $\mathrm{Spin}(V_K)_P$. If n is even, then the stabilizer has two connected components and the identity component is $\mathrm{Spin}(V_K)_P$. In particular, w determines P and $\mathbb{P}(P)$ is the unique secant to the spinor variety through w.

Let d be a rational number, such that $-d$ is not a square of a rational number. Set $K := \mathbb{Q}[\sqrt{-d}]$. Let $\sigma : K \to K$ be the involution in $Gal(K/\mathbb{Q})$. Denote by σ also the induced involution on S_K^+, S_K^-, and V_K. Let $Nm : K \to \mathbb{Q}$ be the norm map $Nm(\lambda) = \lambda\sigma(\lambda)$. Denote the group of rational similarities of $V_\mathbb{Q}$ by

$$\tilde{O}(V_\mathbb{Q}) := \{g \in GL(V_\mathbb{Q}) \ : \ (g(v_1), g(v_2)) = c(v_1, v_2), \text{ for some } c \in Nm(K^\times)\}. \tag{2.2.3}$$

Assume that $P \subset S_\mathbb{Q}^+$ is a non-isotropic 2-dimensional subspace, which intersects the spinor variety in $\mathbb{P}(S_K^+)$ in two σ-conjugate points ℓ_1 and ℓ_2 corresponding to two maximal isotropic subspaces W_1 and W_2 of V_K. The vanishing $W_1 \cap W_2 = (0)$ holds, by Lemma 2.2.1.

The subset $V_\mathbb{Q}$ of V_K is equal to $\{v_1 + \sigma(v_1) \ : \ v_1 \in W_1\}$. Given $\lambda \in K^\times$, let $\tilde{e}_\lambda : V_K \to V_K$ act on W_1 by multiplication by λ and on W_2 by multiplication by $\sigma(\lambda)$. Then $\tilde{e}_\lambda$ leaves $V_\mathbb{Q}$ invariant and we get the homomorphism

$$\tilde{e} : K^\times \to GL(V_\mathbb{Q}) \tag{2.2.4}$$

sending λ to the restriction of $\tilde{e}_\lambda$ to $V_\mathbb{Q}$.

Lemma 2.2.4. *The centralizer of $\rho(\mathrm{Spin}(V_\mathbb{Q})_P)$ in $\tilde{O}(V_\mathbb{Q})$ is $\tilde{e}(K^\times)$.*

Proof. The image of $\tilde{e}$ clearly centralizes $\rho(\mathrm{Spin}(V_\mathbb{Q})_P)$. Given $v_1, v_1' \in W_1$, set $v = v_1 + \sigma(v_1)$ and $v' = v_1' + \sigma(v_1')$. Then

$$\begin{aligned}(\tilde{e}_\lambda(v), \tilde{e}_\lambda(v'))_V &= (\lambda v_1 + \sigma(\lambda)\sigma(v_1), \lambda v_1' + \sigma(\lambda)\sigma(v_1'))_V \\ &= Nm(\lambda)\left[(v_1, \sigma(v_1'))_V + (v_1', \sigma(v_1))_V\right] = Nm(\lambda)(v, v').\end{aligned}$$

Conversely, if $g \in \tilde{O}(V_\mathbb{Q})$ centralizes $\rho(\mathrm{Spin}(V_\mathbb{Q})_P)$, then W_1 and W_2 are g invariant and g acts on W_i via multiplication by a scalar $\lambda_i \in K^\times$. Given $v_1 \in W_1$, we have

$$\lambda_1 v_1 = g(v_1) = (\sigma g \sigma)(v_1) = \sigma(\lambda_2 \sigma(v_1)) = \sigma(\lambda_2) v_1.$$

Hence, $\lambda_2 = \sigma(\lambda_1)$ and $g = \tilde{e}(\lambda_1)$. □

Let $V_\mathbb{C} := V^{1,0} \oplus V^{0,1}$ be the Hodge decomposition with respect to the complex structure of $X \times \hat{X}$. We call $\oplus_{p=0}^n H^{p,p}(X, \mathbb{Q})$ the *Hodge ring* of X.

Remark 2.2.5. Note that P is contained in the Hodge ring of X for every complex structure $I : V_\mathbb{R} \to V_\mathbb{R}$ of X, which belongs to $\rho(\mathrm{Spin}(V_\mathbb{R})_P)$.

Lemma 2.2.6. *If P is contained in the Hodge ring, then $W_1^{1,0} := W_{1,\mathbb{C}} \cap V^{1,0}$ and $W_2^{1,0} := W_{2,\mathbb{C}} \cap V^{1,0}$ are both n-dimensional. Equivalently, $W_1^{0,1} := W_{1,\mathbb{C}} \cap V^{0,1}$ and $W_2^{0,1} := W_{2,\mathbb{C}} \cap V^{0,1}$ are both n-dimensional.*

Proof. The two statements are the complex conjugate of each other and are thus equivalent. The isomorphism $m : C(V) \to \mathrm{End}(S)$, given in (2.1.3), endows $C(V)$ with an integral Hodge structure. The subspace V of $C(V)$ is a sub-Hodge-structure, which agrees with the natural Hodge structure of V, provided we adjust the weight of the direct summand $H^1(\hat{X},\mathbb{Z})$ of V to be -1 as is the weight of the Hodge structure dual to $H^1(X,\mathbb{Z})$. With this weight adjusment the homomorphism $V \to \mathrm{End}(S)$, given in (2.1.2), becomes a morphism of Hodge structures. Let λ_i be a non-zero element of $\tilde{\ell}_i$, $i = 1, 2$. Assume that P is contained in the Hodge ring. Then λ_i belongs to $\oplus_{p=0}^n H^{p,p}(X)$, for $i = 1, 2$. Hence,

$$m_{\lambda_i} : V_{\mathbb{C}} \to S^-_{\mathbb{C}}$$

is equivariant with respect to the $U(1)$ (circle) action defined by the Hodge structures of V and S^-. Thus, its kernel W_i is $U(1)$-invariant. The weight adjustment does not change the weights of the $U(1)$-action[4], and so considering V with its original weight one Hodge structure we have $W_{i,\mathbb{C}} = W_{i,\mathbb{C}} \cap V^{1,0} \oplus W_{i,\mathbb{C}} \cap V^{0,1}$. It remains to show that the two direct summands have the same dimension, i.e., that $\wedge^{2n} W_{i,\mathbb{C}}$ is the trivial $U(1)$ character.

We recall next the isomorphism

$$\varphi : S \otimes_{\mathbb{Z}} S \to C(V)$$

of [Ch, III.3.1] and verify that it is an isomorphism of Hodge structure. The product $C(V) \otimes C(V) \to C(V)$ is a morphism of Hodge structures, as such is the product in $\mathrm{End}(S)$. The exterior algebras[5] $S_X := \wedge^* H^1(X,\mathbb{Z})$ and $S_{\hat{X}} := \wedge^* H^1(\hat{X},\mathbb{Z})$ both embed naturally as subalgebras and sub-Hodge-structures of $C(V)$, sending exterior products of elements of $H^1(X,\mathbb{Z})$ to the corresponding products in $C(V)$, since $H^1(X,\mathbb{Z})$ and $H^1(\hat{X},\mathbb{Z})$ are isotropic subspaces. We can thus regard the class $[pt_{\hat{X}}] \in H^{2n}(\hat{X},\mathbb{Z})$, Poincaré dual to a point, as an element of $C(V)$. Define φ by

$$\varphi(u \otimes v) := u[pt_{\hat{X}}]\tau(v), \tag{2.2.5}$$

for all $u, v \in S_X$. The element $[pt_{\hat{X}}]$ of $C(V)$ is a Hodge class (of weight $(-n,-n)$ under the above convention), and the involution τ of S_X is an automorphism of Hodge structure, so φ is an integral homomorphism of Hodge structures. Tensoring with $\mathbb{Q}$ it becomes an isomorphism of $\mathrm{Spin}(V_{\mathbb{Q}})$ representations, by [Ch, III.3.1]. Hence, φ is an injective homomorphism of $\mathrm{Spin}(V)$-representations. Working over $\mathbb{Z}$, the composition $m \circ \varphi : S \otimes_{\mathbb{Z}} S \to \mathrm{End}(S)$ is proved surjective[6] in the proof of [GLO, Prop. 3.2.1(e)]

[4]The weight of the $U(1)$-action on $V^{p,q}$ is $p-q$, and the weight adjustment changes the bidegree $(1,0)$ of $H^{1,0}(\hat{X})$ to bidegree $(0,-1)$ and the bidegree $(0,1)$ of $H^{0,1}(\hat{X})$ to bidegree $(-1,0)$.

[5]Given a free $\mathbb{Z}$-module M, we denote by $\wedge^k M$ the quotient of the k-th tensor power $M^{\otimes k}$ of M over $\mathbb{Z}$ by the submodule of symmetric tensors and set $\wedge^* M := \oplus_{k\geq 0} \wedge^k M$.

[6]One checks that $m \circ \varphi : S \otimes_{\mathbb{Z}} S \to \mathrm{End}(S)$ is equal to $(-1)^n$ times the isomorphism sending $s \otimes t$ to $s \otimes (t, \bullet)_S$. Indeed, $[pt_{\hat{X}}]\tau(t)x[pt_{\hat{X}}] = (-1)^{\frac{(2n)(2n-1)}{2}}(t,x)_S[pt_{\hat{X}}] = (-1)^n(t,x)_S[pt_{\hat{X}}]$, for all $t, x \in S$. Hence,

$$\varphi(s \otimes t)(x[pt_{\hat{X}}]) = s[pt_{\hat{X}}]\tau(t)x[pt_{\hat{X}}] = (-1)^n(t,x)_S s[pt_{\hat{X}}],$$

for all $s, t \in S$. Now, the inclusion $S \subset C(V)$ described above, composed with right multiplication by $[pt_{\hat{X}}]$ yields an embedding $\eta : S \to C(V)$ as the left ideal $I := C(V)[pt_{\hat{X}}]$ and the left action of that $m : C(V) \to \mathrm{End}(S)$ is surjective. Hence, φ is surjective, as m is an isomorphism. We conclude that φ is an isomorphism of $\mathrm{Spin}(V)$ representations as well as of Hodge structures.

The isomorphism φ maps the line $\tilde{\ell}_i \otimes \tilde{\ell}_i$ in $S_{\mathbb{C}} \otimes S_{\mathbb{C}}$ onto the top exterior product $\wedge^{2n} W_{i,\mathbb{C}}$, considered as a one-dimensional subspace of $C(V_{\mathbb{C}})$, by [Ch, III.3.2]. The $U(1)$ character $\wedge^{2n} W_{i,\mathbb{C}}$ is isomorphic to the trivial $U(1)$ character $\tilde{\ell}_i \otimes \tilde{\ell}_i$, as φ is an isomorphism of Hodge structures. $\square$

The character group $\mathrm{Hom}(K^\times, K^\times)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, generated by id and conjugation. Given a vector space U over $\mathbb{Q}$ and a homomorphism $K \to \mathrm{End}_{\mathbb{Q}}(U)$ we get the decompostion $U_K := U \otimes_{\mathbb{Q}} K = \oplus_{(a,b)\in\mathbb{Z}} U_{a,b}$, where $\lambda \in K^\times$ acts on the subspace $U_{a,b}$ of U_K via $\lambda^a \bar{\lambda}^b$. The subspaces $U_{\{a,b\}} := U_{a,b} \oplus U_{b,a}$ are defined over $\mathbb{Q}$. With this notation $W_1 = V_{1,0}$ and $W_2 = V_{0,1}$. Set $\wedge_{a,b} V := (\wedge^{a+b} V)_{a,b} = \wedge^a W_1 \otimes \wedge^b W_2$.

The exterior algebra $\wedge^* V_{\mathbb{C}}$ inherits two commuting actions of $K^\times$ and of $\mathbb{C}^\times$, where $x + iy \in \mathbb{C}^\times$ acts via $x + yI_{V_{\mathbb{R}}}$. Hence, $\wedge^k V_{\mathbb{C}}$ admits the decomposition

$$\wedge^k V_{\mathbb{C}} = \oplus_{a+b=k}^{p+q=k} \wedge_{a,b}^{p,q} V,$$

where the top weights correspond to the $\mathbb{C}^\times$ action on $V_{\mathbb{R}}$ and the bottom weights to the $K^\times$ action on $V_{\mathbb{Q}}$. With this notation we have $W_1^{1,0} = \wedge_{1,0}^{1,0} V$, $W_1^{0,1} = \wedge_{1,0}^{0,1} V$, $W_2^{1,0} = \wedge_{0,1}^{1,0} V$, and $W_2^{0,1} = \wedge_{0,1}^{0,1} V$. Lemma 2.2.6 states that the latter four are all n dimensional.

Below and everywhere in the paper unless mentioned otherwise, the $\mathrm{Spin}(V)$-action on $\wedge^* V$ is the ρ-action in Diagram (1.3.1) preserving the grading.

Lemma 2.2.7. *The $\mathrm{Spin}(V)_P$-invariant subspace $(\wedge^{2j} V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$ is a subspace of $\wedge^{j,j} V_{\mathbb{C}}$. Furthermore,* $\dim(\wedge^k V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P} = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ 1 & \text{if } 0 \le k \le 4n, \ k \text{ is even, and } k \neq 2n, \\ 3 & \text{if } k = 2n. \end{cases}$
The space $(\wedge^{2n} V_K)^{\mathrm{Spin}(V)_P}$ decomposes as a direct sum of three characters of $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ consisting of $\wedge^{2n} W_1$ isomorphic to $\det_1$, $\wedge^{2n} W_2$ isomorphic to $\det_2$, and the trivial character. For $j \neq n$, $0 \le j \le 2n$, the space $(\wedge^{2j} V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$ is a trivial $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ character.

Note: We will see that the space $(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$ is spanned by a non-degenerate 2-form on V_K^*, and so the subspace $(\wedge^* V_K)^{\mathrm{Spin}(V_K)_{\ell_1,\ell_2}}$ consists of powers of this 2-form (see Equation (2.4.2)).

Proof. It suffices to prove the statement for $0 \le k \le 2n$, as $\wedge^k V_{\mathbb{Q}}$ and $\wedge^{4n-k} V_{\mathbb{Q}}$ are dual $\mathrm{Spin}(V)_P$ representations. For $0 \le k \le 2n$, $\wedge^k W_i$, $i = 1, 2$, are dual irreducible $\mathrm{Spin}(V)_P$ representation. Furthermore, $\wedge^a W_i$ and $\wedge^{2n-a} W_i$ are dual representations. We have an isomorphism of $\mathrm{Spin}(V)_P$ representations $\wedge^k V_K \cong \oplus_{a+b=k} (\wedge^a W_1) \otimes (\wedge^b W_2)$. Hence, $(\wedge^k V)^{\mathrm{Spin}(V)_P}$ is trivial, if k is odd, and

$$(\wedge^{2j} V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P} = \text{ rational points of } (\wedge^j W_1 \otimes \wedge^j W_2)^{\mathrm{Spin}(V)_P},$$

$C(V)$ on I corresponds to the action of $C(V)$ on S via m. Hence, the displayed equation becomes $\eta(m_{\varphi(s\otimes t)}(x)) = (-1)^n (t, x)_S \eta(s)$. The equality $m_{\varphi(s\otimes t)}(x) = (-1)^n (t, x)_S s$ follows.

for $0 < j < n$, and it is 1-dimensional, while

$$(\wedge^{2n}V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P} = \text{rational points of } (\wedge^n W_1 \otimes \wedge^n W_2)^{\mathrm{Spin}(V)_P} \oplus \wedge^{2n}W_1 \oplus \wedge^{2n}W_2,$$

and it is 3-dimensional, as $\wedge^{2n}W_i$ is a trivial character of $SL(W_i)$.

Note that $\wedge^{n,n}_{2n,0}V = (\wedge^{2n}V)_{2n,0} = \wedge^{2n}W_1$ and $\wedge^{n,n}_{0,2n}V = \wedge^{2n}W_2$ are one-dimensional complex conjugate subspaces defined over K, and so $\wedge^{n,n}_{2n,0}V \oplus \wedge^{n,n}_{0,2n}V$ is a 2-dimensional subspace defined over $\mathbb{Q}$. The latter subspace is $\mathrm{Spin}(V_K)_P$-invariant.

It remains to prove that the 1-dimensional subspace $[(\wedge^j W_1) \otimes (\wedge^j W_2)]^{\mathrm{Spin}(V)_P}$ is of Hodge type (j,j), i.e., a subspace of $\wedge^{j,j}_{j,j}V$, for $0 \leq j \leq n$. The subspace $[(\wedge^j W_1) \otimes (\wedge^j W_2)]^{\mathrm{Spin}(V)_P}$ is a one-dimensional $U(1)$-invariant subspace of $(\wedge^j W_1) \otimes (\wedge^j W_2)$ defined over $\mathbb{Q}$, hence it must be the trivial $U(1)$-character.

The representations $\wedge^k W_i$, $i = 1, 2$, are dual irreducible $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-representations. Hence, the triviality of the $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-character $(\wedge^{2j}V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$ for $j \neq n$, $0 \leq j \leq 2n$. ☐

2.3. **The isomorphism $\tilde{\varphi} : S \otimes_{\mathbb{Z}} S \to \wedge^* V$.** Let $B_0 : V \otimes_{\mathbb{Z}} V \to \mathbb{Z}$ be a bilinear pairing, not necessarily symmetric, satisfying $B_0(u,u) = \frac{1}{2}(u,u)_V$. Note that $(u,u)_V$ is even, for every $u \in V$. Our choice of B_0 is as follows. Write $v_i = (w_i, \theta_i)$, $i = 1, 2$, with $w_i \in H^1(X, \mathbb{Z})$ and $\theta_i \in H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$. We set[7] $B_0(v_1, v_2) := \theta_2(w_1)$. Given $x \in V$, define $L_x : \wedge^* V \to \wedge^* V$ by $L_x(u) = x \wedge u$. Given $x \in V$, define $\delta_x : \wedge^* V \to \wedge^* V$ as contraction with $B_0(x, \bullet)$. Set $L'_x := L_x + \delta_x$. Then $(L'_x)^2 = \frac{1}{2}(x,x)_V \cdot \mathbb{1}$, where $\mathbb{1}$ is the identity in $\mathrm{End}(\wedge^* V)$. Hence, the map $x \mapsto L'_x$ extends to an algebra homomorphism

$$\psi' : C(V) \to \mathrm{End}(\wedge^* V),$$

by the universal property of $C(V)$ (see [Ch, II.1.1]). Define

$$\psi : C(V) \to \wedge^* V \tag{2.3.1}$$

by $\psi(x) = \psi'(x) \cdot \mathbb{1}$, where $\mathbb{1} \in \wedge^0 V$ is the unit in $\wedge^* V$. Then ψ is a homomorphism of left $C(V)$-modules.

Let $C(V)_k$ be the additive subgroup of $C(V)$ generated by products of j elements of V, for $j \leq k$. We get the increasing filtration $C(V)_0 \subset C(V)_1 \subset \cdots C(V)_{4n} = C(V)$. Let $F^k(\wedge^* V) := \oplus_{i \leq k} \wedge^i V$. Then $\psi(C(V)_k) \subset F^k(\wedge^* V)$. We get the induced surjective homomorphism

$$\bar{\psi}_k : C(V)_k / C(V)_{k-1} \to \wedge^k V,$$

which is injective, as it induces an isomorphism once we tensor with $\mathbb{Q}$ [Ch, II.1.6]. Hence, ψ is an isomorphism of left $C(V)$-modules. Furthermore, the conjugation action of $\mathrm{Spin}(V)$ on $C(V)$ preserves the filtration $C(V)_k$ and induces an action on the associated graded group and $\bar{\psi}_k$ is $\mathrm{Spin}(V)$-equivariant, where the action on $\wedge^k V$ is the one induced from the representation V [Ch, Sec. 3.3].

Consider the composite isomorphism

$$\tilde{\varphi} := \psi \circ \varphi : S \otimes_{\mathbb{Z}} S \to \wedge^* V, \tag{2.3.2}$$

[7]Our choice yields the equality in Lemma 6.3.1 below. The choice of B_0 used in [Ch, Sec. 3.3] is $B_0(v_1, v_2) := \theta_1(w_2)$

where φ is given in (2.2.5). Conjugating the Spin(V)-action via $\tilde{\varphi}$ we get on $\wedge^* V$ the structure of a Spin(V)-representation. The latter depends on the choice of B_0, but the associated graded action, with respect to the increasing filtration $F^k(\wedge^* V)$ does not (see the discussion following the proof of III.3.1 on page 85 in [Ch]).

Remark 2.3.1. We emphasize that ψ depends on the choice of B_0. Hence so does $\tilde{\varphi}$. Note that the projection $\bar{B}_0$ of B_0 to $\wedge^2 V^* := V^* \otimes V^*/Sym^2(V^*)$ is equal to the projection of $B_0 + (\bullet, \bullet)_V$ and thus satisfies

$$\bar{B}_0((w_1, \theta_1), (w_2, \theta_2)) = \frac{1}{2}(\theta_2(\omega_1) - \theta_1(\omega_2)).$$

The right hand side is $-1/2$ times the alternating form of the Poincare line bundle $\mathcal{P}$ [BL, Th. 2.5.1] and so equal to $\frac{1}{2}c_1(\mathcal{P})$, by [BL, 2.6 (2b)]. The choice of B_0 is done so that the construction is over the integers. Working over $\mathbb{Q}$ one can replace B_0 by $\frac{1}{2}(\bullet, \bullet)_V$ in the above construction. In that case the resulting isomorphism $S_{\mathbb{Q}} \otimes S_{\mathbb{Q}} \rightarrow \wedge^* V_{\mathbb{Q}}$ is Spin(V)-equivariant with respect to the Spin(V)-action on $\wedge^* V_{\mathbb{Q}}$ induced by that on $V_{\mathbb{Q}}$ (and preserving the grading) (see [TT, Th. 1(i)], where $\tilde{\varphi}$ is denoted by E).

Lemma 2.3.2. (1) *The element $\tilde{\varphi}([pt_X] \otimes 1 - (-1)^n 1 \otimes [pt_X])$ belongs to $F^{4n-2}(\wedge^* V)$, but not to $F^{4n-3}(\wedge^* V)$.*
(2) *The element $\tilde{\varphi}([pt_X] \otimes 1 + (-1)^n 1 \otimes [pt_X])$ does not belong to $F^{4n-1}(\wedge^* V)$.*

Proof. (1) It suffices to prove that $\varphi([pt_X] \otimes 1 - (-1)^n 1 \otimes [pt_X])$ belongs to $C(V)_{4n-2}$ but not to $C(V)_{4n-3}$. We have

$$\begin{aligned}
\varphi(1 \otimes [pt_X]) &= 1 \cdot [pt_{\hat{X}}] \cdot \tau([pt_X]) = (-1)^{\frac{2n(2n-1)}{2}} \cdot [pt_{\hat{X}}] \cdot [pt_X] = (-1)^n \cdot [pt_{\hat{X}}] \cdot [pt_X], \\
\varphi([pt_X] \otimes 1) &= [pt_X] \cdot [pt_{\hat{X}}].
\end{aligned}$$

Hence, $\varphi([pt_X] \otimes 1 - (-1)^n 1 \otimes [pt_X]) = [pt_X] \cdot [pt_{\hat{X}}] - [pt_{\hat{X}}] \cdot [pt_X]$. It remains to prove that $[pt_X] \cdot [pt_{\hat{X}}] - [pt_{\hat{X}}] \cdot [pt_X]$ belongs to $C(V)_{4n-2}$ but not to $C(V)_{4n-3}$.

Choose a basis $\{e_1, \dots, e_{2n}\}$ of $H^1(X, \mathbb{Z})$ satisfying $[pt_X] = e_1 \wedge \cdots \wedge e_{2n}$. Let $\{f_1, \dots, f_{2n}\}$ be the dual basis of $H^1(\hat{X}, \mathbb{Z})$. Then $[pt_{\hat{X}}] = f_1 \wedge \cdots \wedge f_{2n}$. We have

$$\begin{aligned}
e_{2n-1}e_{2n}f_{2n}f_{2n-1} &= e_{2n-1}f_{2n-1} - e_{2n-1}f_{2n}e_{2n}f_{2n-1} = e_{2n-1}f_{2n-1} - f_{2n}e_{2n-1}f_{2n-1}e_{2n} \\
&= f_{2n-1}f_{2n}e_{2n}e_{2n-1} + [e_{2n-1}f_{2n-1} + e_{2n}f_{2n}] - 1.
\end{aligned}$$

The terms above commute in $C(V)$ with $\{e_1, \dots, e_{2n-2}\}$ and $\{f_1, \dots, f_{2n-2}\}$. Hence,

$$\begin{aligned}
&(e_1e_2 \cdots e_{2n})(f_{2n}f_{2n-1} \cdots f_1) = \\
&(e_1e_2 \cdots e_{2n-2})(f_{2n-2}f_{2n-3} \cdots f_1)\{f_{2n-1}f_{2n}e_{2n}e_{2n-1} + [e_{2n-1}f_{2n-1} + e_{2n}f_{2n}] - 1\} = \\
&\prod_{k=1}^n \{f_{2k-1}f_{2k}e_{2k}e_{2k-1} + [e_{2k-1}f_{2k-1} + e_{2k}f_{2k}] - 1\}.
\end{aligned}$$

We see that the commutator $[pt_X][pt_{\hat{X}}] - [pt_{\hat{X}}][pt_X]$ belongs to $C(V)_{4n-2}$ and its projection to $C(V)_{4n-2}/C(V)_{4n-3}$ maps to $\wedge^{4n-2}V$ as the element

$$2\sum_{k=1}^n \wedge_{j=1, j\neq k}^n (f_{2j-1} \wedge f_{2j} \wedge e_{2j} \wedge e_{2j-1}) \wedge [e_{2k-1} \wedge f_{2k-1} + e_{2k} \wedge f_{2k}].$$

The above is a sum of $2n$ linearly independent terms, hence it does not vanish.

(2) Clear from the above computation. $\square$

The Künneth theorem interprets (2.3.2) as an isomorphism

$$\tilde{\varphi} : H^*(X \times X, \mathbb{Z}) \to H^*(X \times \hat{X}, \mathbb{Z}).$$

2.4. **Polarized abelian varieties of Weil type from oriented K-secant lines.**

Assumption 2.4.1. *The rational plane P is non-isotropic with respect to the pairing (1.2.3) and is contained in the Hodge ring of X. $\mathbb{P}(P)$ intersects the spinor variety in two complex conjugate points defined over $K := \mathbb{Q}[\sqrt{-d}]$, where d is a positive rational number.*

Set

$$f := \tilde{e}_{\sqrt{-d}} : V_{\mathbb{Q}} \to V_{\mathbb{Q}}, \tag{2.4.1}$$

where $\tilde{e}$ is given in (2.2.4) and we choose the square root $\sqrt{-d} := \sqrt{d}\exp(i\pi/2)$ with argument $\pi/2$. The isomorphism f depends on the rational plane P and the choice of the pure spinor ℓ_1 among $\{\ell_1, \ell_2\}$, which is equivalent to a choice of an orientation of P. Note that f belongs to $\tilde{O}(V_{\mathbb{Q}})$, $(f(x), f(y))_V = d(x,y)_V$, and $f^2 = -d$, by Lemma 2.2.4. Hence

$$(f(x), y)_V = \frac{1}{d}(f^2(x), f(y))_V = -(x, f(y))_V,$$

and so f is anti-self-dual. Let $\Xi \in \wedge^2 V_{\mathbb{Q}}^*$ be the 2-form

$$\Xi(x,y) := (f(x), y)_V. \tag{2.4.2}$$

The 2-form Ξ is non-degenerate, since f is invertible and the pairing (1.2.2) on V is non-degenerate. Note the equality

$$\Xi(x,y) = \sqrt{-d}\left((x_1, y_2)_V - (x_2, y_1)_V\right),$$

where $x = x_1 + x_2$ is the decomposition with $x_i \in W_i$, $i = 1, 2$, and $y = y_1 + y_2$ is the analogous decomposition. An element $v \in V_{\mathbb{Q}}$, admits a unique decomposition $v = v_1^{1,0} + v_2^{1,0} + v_1^{0,1} + v_2^{0,1}$, where $v_i^{1,0} \in W_{i,\mathbb{C}} \cap V^{1,0}$ and $v_i^{0,1} \in W_{i,\mathbb{C}} \cap V^{0,1}$, by Lemma 2.2.6. The summands satisfy

$$\begin{aligned} \overline{v_1^{1,0}} &= v_2^{0,1} \\ \overline{v_1^{0,1}} &= v_2^{1,0} \end{aligned}$$

The complex structure $I := I_{V_{\mathbb{R}}}$ is an isometry[8] of $V_{\mathbb{R}}$ satisfying $I^{-1} = -I$, hence I is anti-self-dual with respect to the pairing on $V_{\mathbb{R}}$. The eigenspaces $V^{1,0}$ and $V^{0,1}$ of I in

[8]Let $X = U/\Lambda$ be an n-dimensional compact complex torus, where U is an n-dimensional complex vector space and Λ is a lattice. The complex vector space U is naturally the pair $(H_1(X,\mathbb{R}), I_X)$, where I_X is the complex structure of X. The dual torus $\hat{X}$ is $\mathrm{Hom}_{\bar{\mathbb{C}}}(U, \mathbb{C})/\hat{\Lambda}$, where $\mathrm{Hom}_{\bar{\mathbb{C}}}(U, \mathbb{C})$ consists of $\mathbb{R}$-linear homomorphisms satisfying $\ell(iu) = -i\ell(u)$ and $\hat{\Lambda}$ is the dual lattice with respect to the pairing $\langle \ell, u \rangle := Im\ell(u)$ (see [BL, Sec. 2.4]). Hence, $\langle i\ell, u \rangle = \langle \ell, -iu \rangle$. An element of $\mathrm{Hom}_{\bar{\mathbb{C}}}(U, \mathbb{C})$ is determined by its real part yielding an isomorphism of $H_1(X,\mathbb{R})^*$ with $\mathrm{Hom}_{\bar{\mathbb{C}}}(U, \mathbb{C})$. Under this identification of $H_1(X,\mathbb{R})^*$ with $\mathrm{Hom}_{\bar{\mathbb{C}}}(U, \mathbb{C}) = H_1(\hat{X}, \mathbb{R})$ the complex structure $I_{\hat{X}}$ acts by composing elements of $H_1(X,\mathbb{R})^*$ with $-I_X$. This agrees with the complex structure of the dual Hodge structure. Hence, $(I_X, I_{\hat{X}})$ is an isometry of $H_1(X,\mathbb{R}) \oplus H_1(\hat{X}, \mathbb{R})$ with respect to the pairing $((u_1, \ell_1), (u_2, \ell_2)) = \langle u_1, \ell_2 \rangle + \langle u_2, \ell_1 \rangle$. Now $I_{V_{\mathbb{R}}}$ is the complex structure of the dual Hodge structure, which is again the direct sum of two dual Hodge structures, hence $I_{V_{\mathbb{R}}}$ is an isometry with respect to the pairing (1.2.2).

$V_{\mathbb{C}}$ are thus isotropic. Furthermore, I commutes with f, by Lemma 2.2.6. Hence, Ξ is of Hodge-type $(1,1)$ and $f\circ I$ is self-dual. We get the symmetric bilinear form on $V_{\mathbb{R}}$ given by $g_P(x,y):=\Xi(I(x),y)=(f(I(x)),y)_V$.

Lemma 2.4.2. $g_P(v,v)=2\sqrt{d}(-(v_1^{1,0},v_2^{0,1})_V+(v_1^{0,1},v_2^{1,0})_V)$.

Proof. Our sign convention for square roots yields $\sqrt{-1}\sqrt{-d}=-\sqrt{d}$.

$$\begin{array}{rcl}(f(I(v)),v) & = & -\sqrt{d}(v_1^{1,0}-v_2^{1,0}-v_1^{0,1}+v_2^{0,1},v_1^{1,0}+v_2^{1,0}+v_1^{0,1}+v_2^{0,1})_V\\ & = & -2\sqrt{d}\left((v_1^{1,0},v_2^{0,1})_V-(v_1^{0,1},v_2^{1,0})_V\right),\end{array}$$

where in the second equality we used the fact that W_i, $i=1,2$, $V^{1,0}$, and $V^{0,1}$ are all isotropic with respect to the pairing (1.2.2). □

Remark 2.4.3. The complex structure I on $V_{\mathbb{R}}$ lifts to an element of $\mathrm{Spin}(V_{\mathbb{R}})$. This is a special case of the following more general fact. Let F be a field and let M and N be two complementary maximal isotropic subspaces of V_F. Let $\eta:F^\times\rightarrow SO(V_F)$ send $\lambda\in F^\times$ to $\eta(\lambda)$ acting on L by multiplication by λ^2 and on M by multiplication by λ^{-2}. Then η admits a lift $\tilde{\eta}:F^\times\rightarrow \mathrm{Spin}(V_F)$ defined as follows. Let $\{e_1,\dots,e_{4n}\}$ be a basis of V_F, such that $e_i\in L$, for $1\leq i\leq 2n$, $e_i\in M$, for $2n+1\leq i\leq 4n$, $(e_i,e_j)=1$, if $j=2n+i$ or $i=2n+j$, and $(e_i,e_j)=0$ otherwise. Then

$$\tilde{\eta}(\lambda):=\prod_{k=1}^{2n}(\lambda^{-1}+(\lambda-\lambda^{-1})e_ke_{k+2n})$$

is an element of $\mathrm{Spin}(V_F)$, which maps to $\eta(\lambda)\in SO(V_F)$ (apply the second displayed formula in [I, Sec. 2]). If I is a complex structure on $V_{\mathbb{R}}$ take $L=V^{1,0}$, $N=V^{0,1}$, and $\lambda=e^{\pi i/4}$, to get the element $\tilde{\eta}(e^{i\pi/4})$ of $\mathrm{Spin}(V_{\mathbb{C}})$ which must be already in $\mathrm{Spin}(V_{\mathbb{R}})$ as it maps to $I=\eta(e^{\pi i/4})$ in $SO(V_{\mathbb{R}})$.

Let $\Theta\in H^{1,1}(X,\mathbb{Z})$ be an ample class. Let

$$\theta:H^1(X,\mathbb{Q})^*\rightarrow H^1(X,\mathbb{Q}) \tag{2.4.3}$$

be contraction with Θ and denote its K-linear extension by $\theta:H^1(X,K)^*\rightarrow H^1(X,K)$ and similarly its $\mathbb{R}$ and $\mathbb{C}$-linear extensions. Note that θ is an isomorphism of rational Hodge structures under the identification of $H^1(X,\mathbb{Q})^*$ with $H^1(\hat{X},\mathbb{Q})$. In fact, $-\theta$ is the pullback homomorphism associated to the isogeny $\phi_L:X\rightarrow\hat{X}$ sending x to $\tau_x^*(L)\otimes L^{-1}$, where L is any line bundle on X with Chern class Θ, by[9] [BL, Lemma 2.4.5].

[9] Let H be the hermitian form on the complex vector space $U:=(H_1(X,\mathbb{R}),I_X)$ with imaginary part $-\Theta$. Then the differential ϕ_H of ϕ_L is given by $u\mapsto H(u,\bullet)$, by [BL, Lemma 2.4.5 and 3.6.4]. The pairing between U and $H^1(\hat{X},\mathbb{R}):=\mathrm{Hom}_{\bar{\mathbb{C}}}(U,\mathbb{C})$ is given by $\langle u,\ell\rangle=Im\ell(u)$. Given $u_1,u_2\in U$, we have

$$\langle u_1,\phi_\Theta(u_2)\rangle=ImH(u_2,u_1)=-\Theta(u_2,u_1).$$

Thus, $\phi_H(u_2)=-\Theta(u_2,\bullet)=-\theta(u_2)$. So $-\theta:H^1(X,\mathbb{R})^*\rightarrow H^1(X,\mathbb{R})$ gets identified with the differential ϕ_H of ϕ_L under the identification of its domain $H^1(X,\mathbb{R})^*$ with $H_1(X,\mathbb{R})$ and its codomain $H^1(X,\mathbb{R})$ with $H_1(\hat{X},\mathbb{R})$. Now use the self-duality $\phi_L^*=\phi_L$ [BL, Cor 2.4.6].

Let d be a positive integer. Set $K := \mathbb{Q}[\sqrt{-d}]$. Set $u := \sqrt{-d}\Theta$. Cup product with $\exp(u)$ is a linear automorphism of $H^*(X,K)$ corresponding to the spin representation image of an element $\exp(u)$ of $\mathrm{Spin}(V_K)$ which acts on V_K by

$$\exp(u)\cdot(w,y) = (w-\sqrt{-d}\theta(y), y), \tag{2.4.4}$$

$w \in H^1(X,K)$ and $y \in H^1(\hat{X},K) \cong H^1(X,K)^*$ (see [Ch, III.1.7] and its proof). Note that $\exp(u)$ leaves invariant the pure spinor $[pt] \in H^{2n}(X,K)$ corresponding to the maximal isotropic subspace $H^1(X,K)$ and leaves invariant every element of the latter subspace. On the other hand $\exp(u)$ takes the pure spinor 1 corresponding to $H^1(X,K)^*$ to the class $\exp(u) \in H^{even}(X,K)$. We set $\ell_1 := \mathrm{span}_K\{\exp(u)\}$, $\ell_2 := \mathrm{span}_K\{\exp(\bar{u})\}$, and

$$P := \mathrm{span}\left\{Re(\exp(u)), \frac{Im(\exp(u))}{\sqrt{d}}\right\} \tag{2.4.5}$$

oriented via the choice of ℓ_1. The maximal isotropic subspaces corresponding to ℓ_1 and ℓ_2 are

$$\begin{array}{lclcl} W_1 & := & \exp(u)(H^1(X,K)^*) & = & \{(-\sqrt{-d}\theta(y), y) \ : \ y \in H^1(X,K)^*\}, \\ W_2 & := & \overline{W_1} & = & \{(+\sqrt{-d}\theta(y), y) \ : \ y \in H^1(X,K)^*\}. \end{array} \tag{2.4.6}$$

The intersection $W_1 \cap W_2$ is the zero subspace, since θ is an isomorphism.

Proposition 2.4.4. *The symmetric bilinear pairing g_P of Lemma 2.4.2 associated with the oriented plane P in (2.4.5) is negative definite.*

Proof. Let x be a class in $V_{\mathbb{Q}}$ and write $x = x_1 + x_2$, with $x_i \in W_i$. Then $x_1 = (-\sqrt{-d}\theta(y), y)$, for some $y \in H^1(X,K)^*$, and $x_2 = \bar{x}_1 = (\sqrt{-d}\theta(\bar{y}), \bar{y})$. We have noted that θ, given in (2.4.3), is an isomorphism of rational Hodge structures under the identification $H^1(X,\mathbb{Q})^*$ with $H^1(\hat{X},\mathbb{Q})$. We identify $H^1(X,K)^*$ with $H^1(\hat{X},K)$ and regard it as a subset of $H^1(\hat{X},\mathbb{C})$. Hence, $\theta(y)^{1,0} = \theta(y^{1,0})$ and $\theta(y)^{0,1} = \theta(y^{0,1})$ and so

$$\begin{array}{rcl} x_1^{1,0} & = & (-\sqrt{-d}\theta(y^{1,0}), y^{1,0}), \\ x_1^{0,1} & = & (-\sqrt{-d}\theta(y^{0,1}), y^{0,1}). \end{array}$$

The first equality below follows from Lemma 2.4.2, the second by definition, the third from the fact that the direct summands $H^1(X,\mathbb{C})$ and $H^1(\hat{X},\mathbb{C})$ of $V_{\mathbb{C}}$ are isotropic. The fourth, since $V^{1,0}$ and $V^{0,1}$ are isotropic.

$$\begin{array}{rcl} g_P(x,x) & = & 2\sqrt{d}(-(x_1^{1,0}, \overline{x_1^{1,0}})_V + (x_1^{0,1}, \overline{x_1^{0,1}})_V) \\ & = & 2\sqrt{d}\left[-((-\sqrt{-d}\theta(y^{1,0}), y^{1,0}), (\sqrt{-d}\theta(\overline{y^{1,0}}), \overline{y^{1,0}}))_V \right. \\ & & \left. +((-\sqrt{-d}\theta(y^{0,1}), y^{0,1}), (\sqrt{-d}\theta(\overline{y^{0,1}}), \overline{y^{0,1}}))_V\right] \\ & = & 2di\left[-(y^{1,0}, \theta(\overline{y^{1,0}}))_V + (\theta(y^{1,0}), \overline{y^{1,0}})_V + (y^{0,1}, \theta(\overline{y^{0,1}}))_V - (\theta(y^{0,1}), \overline{y^{0,1}})_V\right] \\ & = & 2di\left[(-y^{1,0}+y^{0,1}, \theta(\bar{y}))_V + (\theta(y^{1,0}-y^{0,1}), \bar{y})_V\right] \\ & = & 2d\left[-(I(y), \theta(\bar{y}))_V + (\theta(I(y)), \bar{y})_V\right] \\ & = & -4d\Theta(\bar{y}\wedge I(y)). \end{array}$$

Write $y = a + ib$, where a, b in $H^1(\hat{X}, \mathbb{R})$. Then $\bar{y} \wedge I(y) = (a - ib) \wedge (I(a) + iI(b)) = a \wedge I(a) + b \wedge I(b) + i[a \wedge I(b) - b \wedge I(a)]$. The fact that Θ is of type $(1,1)$ yields

$$\Theta(a \wedge I(b)) = \Theta(I(a) \wedge I^2(b)) = -\Theta(I(a) \wedge b) = \Theta(b \wedge I(a)).$$

Hence, $\Theta(\bar{y} \wedge I(y)) = \Theta(a \wedge I(a)) + \Theta(b \wedge I(b))$. The two summands are non-negative and the sum is non-zero if $y \neq 0$, as Θ is ample (we use the sign convention of [H1, Lemma 1.2.15]). □

3. A $\mathrm{Spin}(V)_P$-invariant Hermitian form

In Section 3.1 we show that an oriented K-secant P, for a quadratic imaginary number field K, determines a $\mathrm{Spin}(V)_P$-invariant hermitian form on V, up to a rational scalar. We show that the examples of $2n$-dimensional polarized abelian varieties of Weil type, considered in Section 2.4, all have discriminat $(-1)^n$. In Section 3.2 we give a criterion for an element of $\mathrm{Spin}(V)_P$ to act on $V_{\mathbb{R}}$ as a complex structure of an abelian variety of Weil type on $V_{\mathbb{R}}/V$.

3.1. **The Hermitian form.** Keep Assumption 2.4.1. Let $f := \tilde{e}_{\sqrt{-d}} : V_{\mathbb{Q}} \to V_{\mathbb{Q}}$ be the similarity given in Equation (2.4.1). Let

$$SO_+(V_{\mathbb{Q}})_f \tag{3.1.1}$$

be the subgroup of $SO_+(V_{\mathbb{Q}})$ of elements g, which commute with f and which restriction $g_{|W_i}$ to each of the eigenspaces W_i of f satisfy $\det(g_{|W_i}) = 1$, $i = 1, 2$. Let σ be the generator of $Gal(K/\mathbb{Q})$. Let $u_1 \in S^+_K$ be an even pure spinor corresponding to W_1. Then $u_2 := \sigma(u_1)$ is an even pure spinor corresponding to W_2.

Lemma 3.1.1. *The stabilizer* $\mathrm{Spin}(V_{\mathbb{Q}})_P$ *is mapped by* ρ *isomorphically onto* $SO_+(V_{\mathbb{Q}})_f$*.*

Proof. The homomorphism ρ restricts to an injective homomorphism from the stabilizer $\mathrm{Spin}(V_{\mathbb{Q}})_P$ into $SO_+(V_{\mathbb{Q}})_f$, since the kernel of ρ has order two, generated by $-1 \in C(V_{\mathbb{Q}})$, and the latter acts by $-id_{S_{\mathbb{Q}}}$ on the spin representation. Hence, the kernel of ρ intersects $\mathrm{Spin}(V_{\mathbb{Q}})_P$ trivially. It remain to prove its surjectivity. Let g be an element of $SO_+(V_{\mathbb{Q}})_f$. The homomorphism $\rho : \mathrm{Spin}(V_{\mathbb{Q}}) \to SO_+(V_{\mathbb{Q}})$ is surjective, so we may choose an element $\tilde{g} \in \mathrm{Spin}(V_{\mathbb{Q}})$ satisfying $\rho(\tilde{g}) = g$. There exists a $\mathrm{Spin}(V_{\mathbb{Q}})$ equivariant homomorphism $\wedge^{2n}_+ V_K \to \mathrm{Sym}^2(S^+_K)$, from the subspace $\wedge^{2n}_+ V_K$ of $\wedge^{2n} V_K$ spanned by the top exterior powers $\wedge^{2n} W$ of maximal isotropic subspaces W, which maps the line $\wedge^{2n} W$ to the line spanned by the square u^2 of the corresponding even pure spinor u, by [Ch, III.3.2 and III.4.5]. The element $\tilde{g}$ acts on the line spanned by u_i^2 as the identity, since $\wedge^{2n} g$ acts on $\wedge^{2n} W_i$ as the identity, by definition of $SO_+(V_{\mathbb{Q}})_f$. Hence, $\tilde{g}$ acts on the line $[u_i]$ spanned by u_i via multiplication by a scalar λ_i equal to 1 or -1. Now, $u_2 = \sigma(u_1)$ and $\tilde{g} = \sigma \circ \tilde{g} \circ \sigma$, since $\tilde{g}$ is defined over $\mathbb{Q}$. Hence $\lambda_2 = \sigma(\lambda_1) = \lambda_1$. We conclude that one of $\tilde{g}$ or $-\tilde{g}$ acts as the identity on the plane $P = \mathrm{span}\{u_1, u_2\}$ and so belongs to $\mathrm{Spin}(V_{\mathbb{Q}})_P$. □

Consider the K-valued bilinear form on $V_{\mathbb{Q}}$ given by

$$H(x, y) := d(x, y)_V + \sqrt{-d}(f(x), y)_V. \tag{3.1.2}$$

Lemma 3.1.2. *H is an $SO_+(V_{\mathbb{Q}})_f$-invariant Hermitian form on $V_{\mathbb{Q}}$ considered as a K-vector space, i.e., we have $H(x,y)=\sigma(H(y,x))$ and $H(x,\tilde{e}_\lambda(y))=\lambda H(x,y)$, for $\lambda\in K$. The signature of H is (n,n). The group $SO_+(V_{\mathbb{Q}})_f$ is a finite index subgroup of the subgroup $SU(V_{\mathbb{Q}},H)$ of $SL(V_{\mathbb{Q}})$ leaving H invariant.*

Proof. The automorphism f is anti-self-dual with respect to the pairing (1.2.2), $(f(x),y)_V=(x,-f(y))_V$, for all $x,y\in V_{\mathbb{Q}}$, and $f^2=-d\mathbb{1}_{V_{\mathbb{Q}}}$, where $\mathbb{1}_{V_{\mathbb{Q}}}$ is the identity endomorphism of $V_{\mathbb{Q}}$. So, $\tilde{f}:=\frac{1}{\sqrt{d}}f$ is a complex structure and an isometry of $V_{\mathbb{R}}$ and $(x,y)_V+i(\tilde{f}(x),y)$ is a $\mathbb{C}$-valued Hermitian form on $V_{\mathbb{R}}$. Multiplying by $\sqrt{d}$ we see that H, given in (3.1.2), is a K-valued Hermitian form on $V_{\mathbb{Q}}$, considered as a $2n$-dimensional K-vector space. H is $SO_+(V_{\mathbb{Q}})_f$-invariant, since f centralizes $SO_+(V_{\mathbb{Q}})_f$.

The signature of H is (a,b) if the $2n\times 2n$ diagonal Gram matrix G (necessarily with rational entries) of the quadratic form $H(x,x)$ with respect to some orthogonal K-basis $\{v_1,\dots,v_{2n}\}$ of $V_{\mathbb{Q}}$ has a positive and b negative diagonal entries. The quadratic form depends only on the real part of H. Hence, $\{v_1,\dots,v_{2n},f(v_1),\dots,f(v_{2n})\}$ is an orthogonal $\mathbb{Q}$-basis of $V_{\mathbb{Q}}$ with respect to the real part of H and the Gram matrix of the quadratic form is diagonal of the form $\begin{pmatrix} G & 0\\ 0 & dG\end{pmatrix}$. Hence, if the signature of H is (a,b) then that of the latter is $(2a,2b)$. On the other hand, the latter quadratic form is that of d times the bilinear pairing (1.2.2) on $V_{\mathbb{Q}}$, which has signature $(2n,2n)$. Hence, $a=b=n$.

The norm character has finitely many values on $SO(V_{\mathbb{Q}})$, hence it suffices to show that the group $SO(V_{\mathbb{Q}})_f$ of isometries of $(V_{\mathbb{Q}},(\bullet,\bullet)_V)$ commuting with f and restricting to W_i with determinant 1 is equal to $SU(V_{\mathbb{Q}},H)$. The inclusion $SO(V_{\mathbb{Q}})_f\subset SU(V_{\mathbb{Q}},H)$ is clear. It remains to show that the subgroup of $SU(V_{\mathbb{Q}},H)$ of elements, which restrict to W_i with determinant 1, $i=1,2$, is a finite index subgroup. Let $\beta:=\{e_1,\dots,e_{2n}\}$ be a basis of W_1 and set $\sigma(\beta):=\{\sigma(e_1),\dots,\sigma(e_{2n})\}$. Then $\beta\cup\sigma(\beta)$ is a basis of V_K. Let g be an element of $SU(V_{\mathbb{Q}},H)$. Then g commutes with f. Let M be the matrix of $g_{|W_1}$ in the basis β. Then the matrix of $g_{|W_2}$ in the basis $\sigma(\beta)$ is $\sigma(M)$. Now $1=\det(g)$ is equal to the determinant of g as an element of $GL(V_K)$, which is $\det(M)\det(\sigma(M))$. Thus, $\det(M)$ is a unit in the quadratic imaginary number field K. The statement follows, since the number of units is finite. □

Lemma 3.1.3. *Assume that the similarity $f:V_{\mathbb{Q}}\to V_{\mathbb{Q}}$ given in Equation (2.4.1) is defined in terms of the oriented plane P given in Equation (2.4.5). Then the discriminant of the hermitian form H is $(-1)^n$.*

Proof. Given a basis $\{y_1,\dots,y_{2n}\}$ of $H^1(\hat{X},\mathbb{Q})$ we get the K-basis $\{(0,y_1),\dots,(0,y_{2n})\}$ of $V_{\mathbb{Q}}$. We evaluate $\det(H((0,y_i),(0,y_j)))$. Given $y\in H^1(\hat{X},\mathbb{Q})$, we get the element

$\exp(u)\cdot(0,y)=(-\sqrt{-d}\theta(y),y)$ of W_1, by Equation (2.4.4). We get

$$\begin{array}{rcl}(0,y) & = & (1/2)[\exp(u)\cdot(0,y)+\overline{\exp(u)}\cdot(0,y)],\\ 2f(0,y) & = & \sqrt{-d}\exp(u)\cdot(0,y)+(-\sqrt{-d})\overline{\exp(u)}\cdot(0,y)\\ & = & \sqrt{-d}(-\sqrt{-d}\theta(y),y)-\sqrt{-d}(\sqrt{-d}\theta(y),y)=(2d\theta(y),0),\\ H((0,y_i),(0,y_j)) & = & d((0,y_i),(0,y_j))_V+\sqrt{-d}(f(0,y_i),(0,y_j))=d\sqrt{-d}\Theta(y_i,y_j),\\ \det(H(((0,y_i),(0,y_j))) & = & (d\sqrt{-d})^{2n}\det(\Theta(y_i,y_j))=(-1)^n d^{3n}\det(\Theta(y_i,y_j)).\end{array}$$

Now, $\det(\Theta(y_i,y_j))$ is the square of a rational number, since Θ is anti-symmetric, and $d^{3n}=Nm((\sqrt{-d})^{3n})$. Hence, $\det(H(((0,y_i),(0,y_j)))Nm(K^\times)=(-1)^n Nm(K^\times)$. $\square$

3.2. **Elements of** $\mathrm{Spin}(V_{\mathbb{C}})_P$ **which are complex sturctures on abelian varieties of Weil type.** Let $\tilde{I}$ be an element of $\mathrm{Spin}(V_{\mathbb{R}})_P$, such that $I:=\rho(\tilde{I})$ is a complex structure on $V_{\mathbb{R}}$. I belongs to $\rho(\mathrm{Spin}(V_{\mathbb{R}})_P)$ and so it commutes with $f:=\tilde{e}_{\sqrt{-d}}$, by Lemma 3.1.1. Hence, $I\circ f)^2=d\mathbb{1}_{V_{\mathbb{R}}}$. Let $\nu(I)$ be the multiplicity of the positive square root $\sqrt{d}$ as an eigenvalue of $I\circ f$.

Lemma 3.2.1. *Assume that $\nu(I)=2n$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspaces of I in $V_{\mathbb{C}}$ with eigenvalues $\pm\sqrt{-1}$. Then each of $V^{1,0}$ and $V^{0,1}$ intersects each of $W_{1,\mathbb{C}}$ and $W_{2,\mathbb{C}}$ along an n-dimensional subspace. Furthermore, we have*

$$V^{1,0}\cap W_{2,\mathbb{C}}=\left(V^{1,0}\cap W_{1,\mathbb{C}}\right)^{\perp}\cap W_{2,\mathbb{C}}, \tag{3.2.1}$$

where $\left(V^{1,0}\cap W_{1,\mathbb{C}}\right)^{\perp}$ is the subspace orthogonal to $V^{1,0}\cap W_{1,\mathbb{C}}$ with respect to the pairing $(\bullet,\bullet)_V$.

Proof. The elements I and f are simultaneously diagonalizable and

$$(V^{1,0}\cap W_{1,\mathbb{C}})\oplus(V^{1,0}\cap W_{2,\mathbb{C}})\oplus(V^{0,1}\cap W_{1,\mathbb{C}})\oplus(V^{0,1}\cap W_{2,\mathbb{C}})=V_{\mathbb{C}}.$$

The dimension of each direct summand above in n, by Remark 2.2.5 and Lemma 2.2.6 in the special case when I is associated to a complex structure on X and P is contain in the Hodge ring. We provide a proof below for the general case.

The subspace $V^{1,0}\cap W_{1,\mathbb{C}}$ is the simultaneous eigenspace with eigenvalues $\sqrt{-1}$ for I and $\sqrt{-d}$ for f, and we will abbreviate it by saying that it is the $(\sqrt{-1},\sqrt{-d})$-eigenspace. Similarly, the other three summands are the $(\sqrt{-1},-\sqrt{-d})$, $(-\sqrt{-1},\sqrt{-d})$, and $(-\sqrt{-1},-\sqrt{-d})$-eigenspaces. Now $(I\circ f)^2=d\mathbb{1}_{V_{\mathbb{R}}}$ and $I\circ f$ is defined over $\mathbb{R}$. Hence, its eigenspaces L_1 and L_2 in $V_{\mathbb{C}}$, with eigenvalues $-\sqrt{d}$ and $\sqrt{d}$ respectively, are defined over $\mathbb{R}$. Clearly, $L_1=(V^{1,0}\cap W_{1,\mathbb{C}})+(V^{0,1}\cap W_{2,\mathbb{C}})$ and $L_2=(V^{1,0}\cap W_{2,\mathbb{C}})+(V^{0,1}\cap W_{1,\mathbb{C}})$. So $L_1\cap W_{1,\mathbb{C}}=V^{1,0}\cap W_{1,\mathbb{C}}$ and $L_2\cap W_{1,\mathbb{C}}=V^{0,1}\cap W_{1,\mathbb{C}}$ and similarly for $W_{2,\mathbb{C}}$. Finally, $\sigma(L_1\cap W_{1,\mathbb{C}})=L_1\cap W_{2,\mathbb{C}}$ and $\sigma(L_2\cap W_{1,\mathbb{C}})=L_2\cap W_{2,\mathbb{C}}$. The equality $\dim(L_1\cap W_{1,\mathbb{C}})+\dim(L_1\cap W_{2,\mathbb{C}})=\dim(L_1)=2n$ implies that $\dim(L_1\cap W_{1,\mathbb{C}})=\dim(L_1\cap W_{2,\mathbb{C}})=n$ and similarly for L_2.

The equality

$$\left(V^{1,0}\cap W_{1,\mathbb{C}}\right)^{\perp}=(V^{1,0}\cap W_{1,\mathbb{C}})\oplus(V^{1,0}\cap W_{2,\mathbb{C}})\oplus(V^{0,1}\cap W_{1,\mathbb{C}}).$$

holds, since both subspaces in the above equation are $3n$-dimensional and the inclusion of the left hand side in the right hand side follows from the fact that $V^{1,0}$ and $W_{1,\mathbb{C}}$ are both isotropic. Equation (3.2.1) follows from the equality $W_{2,\mathbb{C}} = (V^{1,0}\cap W_{2,\mathbb{C}})+(V^{0,1}\cap W_{2,\mathbb{C}})$ and the one displayed above. □

Corollary 3.2.2. *The plane $\wedge^{2n}W_1+\wedge^{2n}W_2$ in $\wedge^{2n}V_K$ is defined over $\mathbb{Q}$ and consists of rational classes of Hodge-type (n,n) (the Hodge-Weil classes) for every complex structure I in $\rho(\mathrm{Spin}(V_\mathbb{R})_P)$ satisfying $\nu(I)=2n$.*

Proof. $\wedge^{2n}W_1+\wedge^{2n}W_2$ is defined over $\mathbb{Q}$, since it is defined over K and is σ-invariant. Each of $\wedge^{2n}W_i$, $i=1,2$, consists of classes of Hodge-type (n,n), by Lemma 3.2.1. □

Given $x,y\in V_\mathbb{R}$ and $\tilde{I}\in \mathrm{Spin}(V_\mathbb{R})_w$, such that $I:=\rho(\tilde{I})$ is a complex structure on $V_\mathbb{R}$ as in Corollary 3.2.2, set

$$\begin{aligned}\Theta_P(x,y) &:= (f(x),y)_V,\\ g_I(x,y) &:= \Theta_P(x,I(y)) = (f(x),I(y))_V.\end{aligned}$$

The automorphisms I and f of $V_\mathbb{R}$ commute and both are anti-self-dual with respect to the symmetric bilinear pairing $(\bullet,\bullet)_V$ on $V_\mathbb{R}$. Hence, g_I is symmetric:

$$g_I(y,x)=(f(y),I(x))_V=-(y,f(I(x)))_V=-(y,I(f(x)))_V=(I(y),f(x))_V=g_I(x,y).$$

Corollary 3.2.3. *If the bilinear form g_I is positive definite, then the rational $(1,1)$ class $\Theta_P(x,y)$ is a Kähler class, and so the complex torus $(V_\mathbb{R}/V_\mathbb{Z},I,\Theta_P)$ is a polarized abelian variety of Weil type.*

Proof. Set $A:=V_\mathbb{R}/V_\mathbb{Z}$, endowed with the complex structure I. Consider the embedding $K\hookrightarrow \mathrm{End}_\mathbb{Q}(A)$, which sends $\sqrt{-d}$ to f, given in (2.4.1). Then

$$f^*\Theta_P(x,y):=\Theta_P(f(x),f(y))=(f^2(x),f(y))_V=-(f^3(x),y)_V=d(f(x),y)=d\Theta_P(x,y),$$

verifying the condition on the polarization in [vG1, Def. 4.9]. □

4. An adjoint orbit in $\mathrm{Spin}(V_\mathbb{R})_P$ as a period domain of abelian varieties of Weil type

Recall that ρ maps $\mathrm{Spin}(V_\mathbb{R})_P$ isomorphically onto the subgroup $SO_+(V_\mathbb{R})_f$, by Lemma 3.1.1. Let

$$\Omega_P \subset SO_+(V_\mathbb{R})_f \tag{4.0.1}$$

be the subset of elements I, such that I is a complex structure on $V_\mathbb{R}$, the eigenspaces of $f\circ I$ are both $2n$-dimensional, and the bilinear form g_I in Corollary 3.2.3 is positive definite.

Let $\iota:\Omega_P\to Gr(n,W_{1,\mathbb{C}})$ be given by $\iota(I):=V^{1,0}_I\cap W_{1,\mathbb{C}}$. The map ι is well defined, by Lemma 3.2.1.

Lemma 4.0.1. *The map ι is an embedding of Ω_P as a non-empty subset, open in the classical topology, of the Grassmannian $Gr(n,W_{1,\mathbb{C}})$.*

Proof. The map ι is injective. Indeed, if $U=\iota(I)$, then I is the unique complex structure on $V_\mathbb{R}$ with

$$V^{1,0}_I = U\oplus\left[U^\perp\cap W_{2,\mathbb{C}}\right],$$

by Equation (3.2.1), where $U^\perp$ is the subspace of $V_\mathbb{C}$ orthogonal to U with respect to the pairing $(\bullet,\bullet)_V$. Given an n-dimensional subspace U of $W_{1,\mathbb{C}}$, set $V_U^{1,0} := U \oplus \left[U^\perp \cap W_{2,\mathbb{C}}\right]$. Then $V_U^{1,0}$ is an isotropic $2n$-dimensional subspace of $V_\mathbb{C}$, invariant under f. The condition that $V_U^{1,0}$ and its complex conjugate $V_U^{0,1}$ are transversal is open. If U is such, let I_U be the complex structure on $V_\mathbb{R}$ with $V_{I_U}^{1,0} = V_U^{1,0}$. The subspace $V_{I_U}^{1,0}$ is f-invariant and f is defined over $\mathbb{Q}$ and so $V_{I_U}^{0,1}$ is f-invariant as well. Hence, I_U commutes with f. The condition that g_I is positive definite is open, as is the condition that I_U preserves the orientation of the positive cone in $V_\mathbb{R}$. We conclude that I_U belongs to $SO_+(V_\mathbb{R})_f$ for U in an open analytic subset of $Gr(n, W_{1,\mathbb{C}})$. Furthermore, the $-\sqrt{d}$-eigenspace of $I \circ f$ is $U \oplus \bar{U}$ and is $2n$-dimensional. Hence, $\nu(I) = 2n$.

It remains to prove that Ω_P is non-empty. Note that for I the complex structure of $X \times \hat{X}$ the negative definite bilinear form g_P of Proposition 2.4.4 is $-g_I$. Hence, g_I is positive definite. Assumption 2.4.1 and Lemma 2.2.6 imply that I commutes with f. Remark 2.4.3 verifies that I is in $SO_+(V_\mathbb{R})$. The restriction of I to W_i has determinant 1, for $i = 1, 2$, and $\nu(I) = 2n$, by Lemma 2.2.6. Hence, I belongs to $SO_+(V_\mathbb{R})_f$ and so to Ω_P. □

Lemma 4.0.2. *The connected components of Ω_P are $SO_+(V_\mathbb{R})_f$-adjoint orbits.*

Proof. Set $I_1 := hIh^{-1}$, for some $h \in SO_+(V_\mathbb{R})_f$ and $I \in \Omega_P$. Then f commutes with I and h and so

$$\begin{aligned} g_{I_1}(x,y) &= \Theta_P(x, I_1(y)) = (f(x), hIh^{-1}(y))_V = (h^{-1}(f(x)), I(h^{-1}(y)))_V \\ &= (f(h^{-1}(x)), I(h^{-1}(y)))_V = g_I(h^{-1}(x), h^{-1}(y)). \end{aligned}$$

Hence, g_{I_1} is positive definite as well.

It remains to prove that the dimensions of the adjoint orbits of $I \in \Omega_P$ is equal to that of Ω_P. The dimension $\dim_\mathbb{Q}(SO_+(V_\mathbb{Q})_f)$ is the dimension $(2n)^2-1$ of $SU(V_\mathbb{Q}, H)$, by Lemma 3.1.2. Hence, $\dim_\mathbb{R}(SO_+(V_\mathbb{R})_f) = 4n^2-1$. If $g \in SO_+(V_\mathbb{R})_f$ commutes with $I \in \Omega_P$, then g leaves invariant each of the direct summands $W_1^{1,0}$, $W_1^{0,1}$, $W_2^{1,0}$, $W_2^{0,1}$ in Lemma 2.2.6. Then $V_\mathbb{R}$ decomposes as the H-orthogonal direct sum of two g-invariant H-non-degenerate subspaces $[W_1^{1,0} \oplus W_2^{0,1}] \cap V_\mathbb{R}$ and $[W_1^{0,1} \oplus W_2^{1,0}] \cap \mathbb{R}$. The determinants of the restrictions of g to the two direct summands are inverses of each other. We conclude that the real dimension of the commutator of I is $2\dim(U(n)) - 1 = 2n^2 - 1$. Thus the real dimension of the adjoint orbit is $2n^2$. This is the dimension of each component of Ω_P, by Lemma 4.0.1. □

We may regard Ω_P as a union of adjoint orbits in $\mathrm{Spin}(V_\mathbb{R})_P$, by Lemmas 3.1.1 and 4.0.2. Given a complex structure $I \in \Omega_P$, denote by $\tilde{I}$ the unique element in $\mathrm{Spin}(V_\mathbb{R})_P$ satisfying $\rho(\tilde{I}) = I$ (Lemma 3.1.1). Note that $m(\tilde{I})$ need not preserve the grading of $H^*(X,\mathbb{R})$. Consider the subgroup $\mathbb{S}_I := \{\cos(\theta) id_{V_\mathbb{R}} + \sin(\theta) I \ : \ \theta \in \mathbb{R}\}$ of $SO_+(V_\mathbb{R})$. The identity component of its inverse image in $\mathrm{Spin}(V_\mathbb{R})$ defines a real Hodge structure of weight 0 on $S^+_\mathbb{R} = H^{ev}(X,\mathbb{R})$. In other words, we get a decomposition $S^+_\mathbb{C} := \oplus S^{-p,p}_\mathbb{C}$, with p an integer, satisfying $\overline{S^{-p,p}_\mathbb{C}} = S^{p,-p}_\mathbb{C}$. We will refer to a rational class in $S^{0,0}_\mathbb{C}$ as a *semi-Hodge class*.

Lemma 4.0.3. (1) *The plane $\wedge^{2n}W_1 \oplus \wedge^{2n}W_2$ corresponds to a rational plane in $\wedge^{2n}V_{\mathbb{Q}}$, which is spanned by Hodge classes, for every complex structure I in Ω_P.*
(2) *The plane P is spanned by semi-Hodge classes, for every complex structure I in Ω_P.*

Proof. Part (1) is Lemma 3.2.2. (2) It suffices to show that $I_\theta := \cos(\theta)id_{V_{\mathbb{R}}} + \sin(\theta)I$ belongs to $SO_+(V_{\mathbb{R}})_f$, for all $\theta \in \mathbb{R}$, since then the inverse image of $\mathbb{S}_I$ in $\mathrm{Spin}(V_{\mathbb{R}})$ is disconnected, with the identity component in $\mathrm{Spin}(V_{\mathbb{R}})_P$ and the other component is its product with -1 and is disjoint from $\mathrm{Spin}(V_{\mathbb{R}})_P$. Hence, classes in P are invariant by under the identity component determining the Hodge structure on $S_{\mathbb{C}}$. We have seen that I is anti-self dual with respect to $(\bullet,\bullet)_V$. Hence, $I_\theta^\dagger = \cos(\theta)id_V - \sin(\theta)I$ and $I_\theta^\dagger I_\theta = id_V$ and so I_θ belongs to $SO(V_{\mathbb{R}})$, for all $\theta \in \mathbb{R}$, and hence to the connected component $SO_+(V_{\mathbb{R}})$ of I. Clearly, I_θ commutes with f, as I does. Now I_θ acts on $W_i^{1,0}$ by scalar multiplication by $\cos(\theta)+\sqrt{-1}\sin(\theta)$ and on $W_i^{0,1}$ by $\cos(\theta)-\sqrt{-1}\sin(\theta)$ and so the determinant of its restriction to W_i is 1. Hence, I_θ belongs to $SO_+(V_{\mathbb{R}})_f$. □

Corollary 4.0.4. *The classes in $(\wedge^*(V_{\mathbb{Q}}))^{\mathrm{Spin}(V)_P}$ remain of Hodge type for every complex structure in Ω_P*

Proof. A class α in $\wedge^2(V_{\mathbb{Q}})$ is of type $(1,1)$ with respect to a complex structure I, if and only if $I(\alpha)=\alpha$. Hence the statement holds for classes in $\wedge^2(V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$. The subring $(\wedge^*(V_{\mathbb{Q}}))^{\mathrm{Spin}(V)_P}$ consists of the direct sum of the rational 2-dimensional subspace of Hodge-Weil classes in $\wedge^{2n}(V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$ and the space of powers of classes in the one-dimensional $\wedge^2(V_{\mathbb{Q}})^{\mathrm{Spin}(V)_P}$, by Lemma 2.2.7. Hence, the statement follows from Lemma 4.0.3(1). □

5. Equivalences of derived categories

In Section 5.1 we recall that the cohomological action $\Phi^H : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$, of an auto-equivalence Φ of the derived category of an abelian variety X, corresponds to an element of the image of $\mathrm{Spin}(V)$, $V := H^1(X\times\hat{X},\mathbb{Z})$, via the spin representation, and that its image via the vector representation is an an element of $SO^+(V)$ preserving the Hodge structure. In section 5.2 we consider two representations of $\mathrm{Spin}(V)$ on $S = H^*(X,\mathbb{Z})$, the spin representation m, and its conjugate $m^\dagger$ via the main anti-automorphism τ. We recall that the Mukai pairing is invariant with respect to the $m\otimes m$-representation on $S\otimes S$, but the Poincaré pairing is invariant with respect to $m\otimes m^\dagger$. Hence, the latter needs to be considered in the context of composition of correspondences.

5.1. **The cohomological action factor through** $\mathrm{Spin}(V)$**.**

Given two smooth projective varieties X and Y and an object F in the bounded derived category $D^b(X\times Y)$ of coherent sheaves, denote by

$$\Phi_F : D^b(X) \to D^b(Y)$$

the integral functor given by $\Phi_F(\bullet) := R\pi_{Y,*}(F\otimes L\pi_X^*(\bullet))$, where the tensor product is the left derived functor. Denote by

$$\Psi_F : D^b(Y) \to D^b(X)$$

the integral functor given by $\Psi_F(\bullet) := R\pi_{X,*}(F \otimes L\pi_Y^*(\bullet))$. Given a third smooth projective variety Z and an object G in $D^b(Y \times Z)$, the composition $\Phi_G \circ \Phi_F$ is isomorphic to the integral transform Φ_{F*G} given by the convolution

$$F * G := R\pi_{13,*}(L\pi_{12}^*(F) \otimes L\pi_{23}^*G),$$

where π_{ij} is the projection onto the product of the i-th and j-th factors of $X \times Y \times Z$, numbered from left to right. Note that $F * G$ is given by the following formula as well:

$$F * G := R\pi_{14,*}(L\pi_{12}^*(F) \otimes L\pi_{23}^*\mathcal{O}_{\Delta_Y} \otimes L\pi_{34}^*G),$$

where π_{ij} are now projections from $X \times Y \times Y \times Z$ and Δ_Y is the diagonal in $Y \times Y$. Denote by F_R the object $F^\vee \otimes \pi_X^*\omega_X[\dim(X)]$. Then $\Psi_{F_R} : D^b(Y) \to D^b(X)$ is the right adjoint functor of Φ_F and is an inverse, if Φ_F is an equivalence. If Φ_F is an equivalence, then so is Φ_{F_R} [H2, Rem. 7.7]. Given an object E in $D^b(X)$ Grothendieck-Verdier-Duality yields the isomorphism

$$\Phi_{F_R}(E) \cong \left(\Phi_F(E^\vee)\right)^\vee, \tag{5.1.1}$$

where $E^\vee$ is the derived dual of E.

Let X be an abelian variety. The group $\mathrm{Aut}(D^b(X))$, of auto-equivalences, fits in the exact sequence

$$0 \to 2\mathbb{Z} \times X \times \hat{X} \to \mathrm{Aut}(D^b(X)) \stackrel{ch}{\to} \mathrm{Spin}_{Hdg}(V_X) \to 0, \tag{5.1.2}$$

where $\mathrm{Spin}_{Hdg}(V_X)$ is the subgroup of $\mathrm{Spin}(V_X)$ preserving the Hodge structure of V_X, and the homomorphism ch sends an auto-equivalence Φ_G with kernel $G \in D^b(X \times X)$ to the homomorphism $\phi_G : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$, given by

$$\phi_G(\bullet) := \pi_{2,*}(\pi_1^*(\bullet) \cup ch(G)),$$

[GLO, Prop. 4.3.7 and Cor. 4.3.8]. An even integer $2k$ in the factor $2\mathbb{Z}$ of the kernel of ch corresponds to an even shift functor mapping an object F to $F[2k]$. The factor X in the kernel of ch acts via translations and the factor $\hat{X}$ via tensorization by line bundles. Equality (5.1.1) corresponds on the cohomological level to the equality

$$\phi_{G_R} = \tau \circ \phi_G \circ \tau, \tag{5.1.3}$$

where τ is the involution given in (1.2.3). We will see below in Equation (5.2.2) that ϕ_{G_R} is the adjoint of ϕ_G^{-1} with respect to the Poincaré pairing. In contrast, the cohomological action of Ψ_{G_R} is the adjoint of ϕ_G with respect to the Mukai pairing, as the inverse is the adjoint of an isometry and ϕ_G is an isometry with respect to the Mukai pairing.

Assume next that X, Y, and Z are n dimensional abelian varieties. Then the canonical line bundle and the Todd classes are all trivial and so

$$ch(F * G) = \pi_{14,*}(\pi_{12}^*ch(F) \cup \pi_{23}^*ch(\mathcal{O}_{\Delta_Y}) \cup \pi_{34}^*ch(G)).$$

In other words, it is the contraction of the two middle factors in the Künneth decomposition of $H^*(X \times Y \times Y \times Z, \mathbb{Q})$ via the Poincaré pairing $H^*(Y, \mathbb{Q}) \otimes H^*(Y, \mathbb{Q}) \to \mathbb{Q}$, given by $(s, t) \mapsto \int_Y s \cup t$. The latter pairing is not $\mathrm{Spin}(V_Y)$-invariant, where $V_Y := H^1(Y, \mathbb{Z}) \oplus H^1(\hat{Y}, \mathbb{Z})$. Consequently, the cohomological convolution is not invariant with respect to the diagonal $\mathrm{Spin}(V_Y)$ action on the two middle factors in the tensor product $S_X \otimes S_Y \otimes S_Y \otimes S_Z$ of the spin representations.

5.2. Spin(V)**-equivariance of convolutions.** The Spin(V_Y)-invariance is restored by changing the Spin(V_Y)-action on the right Künneth factor of $H^*(X\times Y)\cong H^*(X)\times H^*(Y)$ or on the left Künneth factor of $H^*(Y\times Z)$, as we will show in Corollary 5.2.2 below. Let $m:\mathrm{Spin}(V)\rightarrow GL(S)$ be the Spin representation and denote by

$$m^\dagger:\mathrm{Spin}(V)\rightarrow GL(S) \tag{5.2.1}$$

the homomorphism $m^\dagger_g:=\tau m_g\tau$, where τ is given in (1.2.3). We have

$$\int_X m^\dagger_g(s)\cup m_g(t)=(m_g(\tau(s)),m_g(t))_S=(\tau(s),t)_S=\int_X s\cup t, \tag{5.2.2}$$

for all $s,t\in S$ and $g\in\mathrm{Spin}(V)$, where the second equality follows from the Spin(V)-invariance of the Mukai pairing (1.2.3). We conclude that the Poincaré pairing is invariant[10] with respect to the action of Spin(V) on $S\otimes S$ by $g\mapsto m^\dagger_g\times m_g$. Consequently, the Spin-equivariance of the convolution is restored if we let $\mathrm{Spin}(V_X)\times\mathrm{Spin}(V_Y)$ act on $H^*(X\times Y)=S_X\otimes S_Y$ via $m\otimes m^\dagger$ and similarly let $\mathrm{Spin}(V_Y)\times\mathrm{Spin}(V_Z)$ act on $H^*(Y\times Z)$ via $m\times m^\dagger$. Note that the action on the product $Y\times Y$ of the second and third factors of $X\times Y\times Y\times Z$ is via $m^\dagger\times m$ explaining the computation (5.2.2).

Observe that the following analogue of equality (5.2.2) holds as well.

$$\int_X m_g(s)\cup m^\dagger_g(t)=\int_X s\cup t, \tag{5.2.3}$$

for all classes $s,t\in H^*(X)$ and all $g\in\mathrm{Spin}(V)$. Indeed, if s and t are both even, then so are $m_g(s)$ and $m^\dagger_g(t)$ and the above follows from (5.2.2) and the symmetry of the Poincaré pairing on even cohomology, if s and t are both odd, then so are $m_g(s)$ and $m^\dagger_g(t)$ and

$$\int_X s\cup t=-\int_X t\cup s\stackrel{(5.2.2)}{=}-\int_X m^\dagger_g(t)\cup m_g(s)=\int_X m_g(s)\cup m^\dagger_g(t),$$

and if one is odd and the other even, then both sides of (5.2.3) vanish.

Let $PD:H^*(X,\mathbb{Q})\rightarrow H^*(X,\mathbb{Q})^*$ be given by $PD(s):=\int_X(\bullet\cup s)$. Equation (5.2.2) yields $PD(m_h(s))=PD(s)\circ m^\dagger_{h^{-1}}$, for all $h\in\mathrm{Spin}(V)$.

$$\begin{tikzcd} H^*(X) \arrow[r, "PD"] \arrow[d, "m_h"'] & H^*(X)^* \arrow[d, "(\bullet)\circ m^\dagger_{h^{-1}}"] \\ H^*(X) \arrow[r, "PD"'] & H^*(X)^*. \end{tikzcd}$$

Given a class $\gamma\in H^*(X\times Y,\mathbb{Q})$, let $\gamma_*:H^*(X)\rightarrow H^*(Y)$ be the associated correspondence homomorphism $\gamma_*(\bullet):=\pi_{Y,*}(\pi_X^*(\bullet)\cup\gamma)$.

Lemma 5.2.1. *The following equalities hold for all $h\in\mathrm{Spin}(V_X)$ and $g\in\mathrm{Spin}(V_Y)$.*

$$\begin{aligned}[(m_h\otimes m^\dagger_g)(\gamma)]_* &= m^\dagger_g\circ\gamma_*\circ m^\dagger_{h^{-1}},\\ [(m^\dagger_h\otimes m_g)(\gamma)]_* &= m_g\circ\gamma_*\circ m_{h^{-1}}.\end{aligned} \tag{5.2.4}$$

[10]Equivalently, $m^\dagger_{g^{-1}}$ is the left adjoint of m_g with respect to the super-symmetric Poincaré pairing.

Note that Equation (5.2.4) establishes the $\mathrm{Spin}(V_X)\times\mathrm{Spin}(V_Y)$-equivariance of the map $H^*(X\times Y,\mathbb{Q})\rightarrow \mathrm{Hom}(H^*(X,\mathbb{Q}),H^*(Y,\mathbb{Q}))$ sending γ to γ_* with respect to the $m^\dagger\times m$ on the domain and the usual action on the target.

Proof. We prove the first equality. Let $\{u_i\}$ be a basis of $H^*(X,\mathbb{Q})$ and $\{v_j\}$ a basis for $H^*(Y,\mathbb{Q})$. Write $\gamma:=\sum a_{ij}u_i\otimes v_j$. Then $\gamma_*(\bullet)=\sum a_{ij}PD(u_i)\otimes v_j$. We get

$$[(m_h\otimes 1)(\gamma)]_*=\sum a_{ij}PD(m_h(u_i))\otimes v_j \stackrel{(5.2.2)}{=}\sum a_{ij}(PD(u_i)\circ m^\dagger_{h^{-1}})\otimes v_j=\gamma_*\circ m^\dagger_{h^{-1}}(\bullet).$$

The identity $(1\otimes m_g^\dagger)(\gamma)=m_g^\dagger\circ\gamma_*$ is evident, for all $g\in\mathrm{Spin}(V_Y)$.

The proof of the second equality (5.2.4) is identical, using Equation (5.2.3) instead of (5.2.2). □

Let $\gamma\in H^*(X\times Y,\mathbb{Q})$ and $\delta\in H^*(Y\times Z,\mathbb{Q})$. The following $\mathrm{Spin}(V_Y)$-invariance of composition of correspondence homomorphisms is an immediate corollary of Lemma 5.2.1.

Corollary 5.2.2. *The following equalities hold for all $h\in\mathrm{Spin}(V_Y)$.*

$$\begin{aligned}[(m_h\otimes 1)(\delta)]_*\circ[(1\otimes m_h^\dagger)(\gamma)]_* &= \delta_*\circ\gamma_*,\\ [(m_h^\dagger\otimes 1)(\delta)]_*\circ[(1\otimes m_h)(\gamma)]_* &= \delta_*\circ\gamma_*,\end{aligned}$$

Remark 5.2.3. The action of τ on $S^+_{\mathbb{Q}}:=H^{even}(X,\mathbb{Q})$ takes the Chern character $w:=ch(F)$ of an object F in $D^b(X)$ to the Chern character $w^\vee:=ch(F^\vee)$ of the dual object $F^\vee:=R\mathcal{H}om(F,\mathcal{O}_X)$. Hence, $w^\vee$ is $\mathrm{Spin}(V)_w$-invariant with respect to the $m^\dagger$-action.

6. Orlov's derived equivalence $\Phi:D^b(X\times X)\rightarrow D^b(X\times\hat{X})$

In Section 6.1 we recall the definition of Orlov's equivalence $\Phi:D^b(X\times X)\rightarrow D^b(X\times\hat{X})$ and reduce the proof of its main $\mathrm{Spin}(V)$-equivariance property to the surface case. In Section 6.2 we complete the proof of the $\mathrm{Spin}(V)$-equivariance property of Φ in the abelian surface case. We also relate Orlov's equivalence to the construction of the $\mathrm{Spin}(7)$-invariant Cayley class in $H^4(X\times\hat{X},\mathbb{Z})$, when X is an abelian surface, used in [M2] to prove the main results of this paper for abelian fourfolds of Weil type. In Section 6.3 we relate the cohomological isomorphism induced by Orlov's equivalence to an isomorphism $S\otimes S\cong\wedge^*V$ constructed by Chevalley. In Section 6.4 we consider a K-secant $P\subset\mathbb{P}(H^*(X,\mathbb{Q}))$ and use Orlov's equivalence to relate the Hodge-Weil classes in the middle cohomology of $X\times\hat{X}$, associated to the complex multiplication $\eta_P:K\rightarrow\mathrm{End}_{\mathbb{Q}}(X\times\hat{X})$, to the plane in $H^*(X\times X,\mathbb{Q})$ spanned by the tensor squares of the two pure spinnors in P.

6.1. $\mathrm{Spin}(V)$**-Equivariance properties of Orlov's equivalence.** Let X be an abelian variety. Let $\mathcal{P}$ be the normalized Poincaré line bundle over $X\times\hat{X}$. Set $n:=\dim_{\mathbb{C}}(X)$. Let $\mu:X\times X\rightarrow X\times X$ be given by $(x_1,x_2)=(x_1+x_2,x_2)$. Let $id\times\Phi_{\mathcal{P}}:D^b(X\times\hat{X})\rightarrow D^b(X\times X)$ be the integral transform with kernel $\mathcal{O}_{\Delta_X}\boxtimes\mathcal{P}$. Then

$$\mu_*\circ(id\times\Phi_{\mathcal{P}}):D^b(X\times\hat{X})\rightarrow D^b(X\times X) \tag{6.1.1}$$

is an equivalence of categories which intertwines the action of $\mathrm{Aut}(D^b(X))$ on $D^b(X\times\hat{X})$ and $D^b(X\times X)$, where an autoequivalence $\Phi_{\mathcal{G}}$ with kernel $\mathcal{G}\in D^b(X\times X)$ acts on $D^b(X\times X)$ via $\Phi_{\mathcal{G}}\times\Phi_{\mathcal{G}^\vee[n]}$, by [H2, Cor. 9.37]. The action of $\Phi_{\mathcal{G}}$ on $D^b(X\times\hat{X})$ is the composition of push-forward via the automorphism associated to the element $\phi_{\mathcal{G}}$ of $\mathrm{Spin}(V_X)$ preserving the Hodge structure, followed by tensorization by a line bundle on $X\times\hat{X}$. The inverse

$$\Phi := (id\times\Psi_{\mathcal{P}^{-1}[n]})\circ\mu^* : D^b(X\times X)\rightarrow D^b(X\times\hat{X}) \tag{6.1.2}$$

intertwines the two actions as well. The above functor is induced by an object in $D^b([X\times X]\times[X\times\hat{X}])$, whose Chern character yields a correspondence isomorphism

$$\phi : S\otimes S\rightarrow \wedge^\bullet V \tag{6.1.3}$$

on the level of cohomology. Note that $\Psi_{\mathcal{G}^\vee[n]}$ is the quasi-inverse of $\Phi_{\mathcal{G}}$, while $\Phi_{\mathcal{G}^\vee[n]}$ is the conjugate of $\Phi_{\mathcal{G}}$ by the dualization functor from $D^b(X)$ to $D^b(X)^{op}$ (see Equation (5.1.1)). Cohomologically, $\phi_{\mathcal{G}^\vee[n]} : S\rightarrow S$ satisfies $\phi_{\mathcal{G}^\vee[n]}=\tau\phi_{\mathcal{G}}\tau$, by (5.1.3).

Let $\psi_{\mathcal{P}^{-1}[n]} : H^*(X,\mathbb{Z})\rightarrow H^*(\hat{X},\mathbb{Z})$ be the cohomological correspondence associated to $\Psi_{\mathcal{P}^{-1}[n]}$. Let $\tilde{\varphi} : H^*(X\times X,\mathbb{Z})\rightarrow H^*(\hat{X}\times X,\mathbb{Z})$ be the isomorphism (2.3.2). The following statement is proved in Section 6.3.

Lemma 6.1.1. $\phi=(\phi_{\mathcal{P}}\otimes\psi_{\mathcal{P}^{-1}[n]})\circ\tilde{\varphi}\circ(id\otimes\tau)$.

Given $g\in\mathrm{Spin}(V)$, the composition

$$\rho'_g := \phi(m_g\times m_g^\dagger)\phi^{-1} : \wedge^\bullet V\rightarrow\wedge^\bullet V \tag{6.1.4}$$

does not preserve the grading, but it preserves the decreasing filtration by the subspaces

$$F_k(\wedge^\bullet V) := \oplus_{i\geq k}\wedge^i V, \tag{6.1.5}$$

and the induced graded action is the one induced from $\rho:\mathrm{Spin}(V)\rightarrow SO(V)$, by [GLO, Prop. 4.3.7 and Cor. 4.3.8], and more directly by Lemma 6.1.1. We thus get two $\mathrm{Spin}(V)$ actions on $\wedge^* V$

$$\rho : \mathrm{Spin}(V) \rightarrow GL(\wedge^* V), \tag{6.1.6}$$

$$\rho' : \mathrm{Spin}(V) \rightarrow GL(\wedge^* V), \tag{6.1.7}$$

where ρ'_g is given in (6.1.4) and ρ_g acts on $\wedge^k V$ by $\wedge^k\rho_g$ and the latter ρ_g is the image of g via $\rho:\mathrm{Spin}(V)\rightarrow SO(V)$.

Given $g\in\mathrm{Spin}(V)$, there exists a topological complex line-bundle N_g on $X\times\hat{X}$, such that

$$\rho'_g = ch(N_g)\cup\rho_g, \tag{6.1.8}$$

by [Or, Theorem 2.10] (see also [H2, Prop. 9.39]). The equality

$$c_1(N_{g_1g_2}) = c_1(N_{g_1})+\rho_{g_1}(c_1(N_{g_2})), \tag{6.1.9}$$

holds for all $g\in\mathrm{Spin}(V)$, by [H2, Exercise 9.41]. The above equality means that the function $c_1(N_{(\bullet)}) : \mathrm{Spin}(V)\rightarrow H^2(X\times\hat{X},\mathbb{Z})$ is a 1-cocycle determining a class in the first group cohomology $H^1(\mathrm{Spin}(V), H^2(X \times \hat{X}, \mathbb{Z}))$. The latter is[11] the extension class of the short exact sequence of $\mathrm{Spin}(V)$-modules

$$0 \to H^2(X \times \hat{X}, \mathbb{Z}) \to H^{ev}(X \times \hat{X}, \mathbb{Z})/F_4(X \times \hat{X}, \mathbb{Z}) \to H^0(X \times \hat{X}, \mathbb{Z}) \to 0,$$

where $F_k(X \times \hat{X}, \mathbb{Z})$, $k \geq 0$, is the decreasing weight filtration (6.1.5) of cohomology preserved by the ρ'-representation (6.1.7) of $\mathrm{Spin}(V)$.

Proposition 6.1.2. *The following equality holds*

$$\rho'_g = \exp\left(\frac{1}{2}[c_1(\mathcal{P}) - \rho_g(c_1(\mathcal{P}))]\right) \cup \rho_g. \tag{6.1.10}$$

Notice that the integral alternating bilinear form $c_1(\mathcal{P})$ and the symmetric bilinear pairing $(\bullet, \bullet)_V$ agree modulo 2, and so the class $\frac{1}{2}[c_1(\mathcal{P}) - \rho_g(c_1(\mathcal{P}))]$ is indeed integral, for every element $g \in \mathrm{Spin}(V)$. The statement of the proposition is motivated by Remark 2.3.1 (its relevance to the isomorphism ϕ is explained by Lemma 6.1.1).

Proof. The special case of the statement, when X is an abelian surface, is proved in Lemma 6.2.5 below. We prove that the general case follows from the surface case.

Step 1: If X_1 and X_2 are abelian varieties and $X = X_1 \times X_2$ and the conjecture is known for X, then it follows for X_i, since it holds for the image of $\mathrm{Spin}(V_{X_1}) \times \mathrm{Spin}(V_{X_2})$ in $\mathrm{Spin}(V_X)$ via the identification of V_X with $V_{X_1} \oplus V_{X_2}$. Indeed, the Poincaré line bundle $\mathcal{P}_X$ is identified with $\mathcal{P}_{X_1} \boxtimes \mathcal{P}_{X_2}$ under the natural isomorphism $X \times \hat{X} \cong [X_1 \times \hat{X}_1] \times [X_2 \times \hat{X}_2]$. It suffices to prove the conjecture for one abelian variety in each dimension, hence for powers E^n of an elliptic curve E, as it is topological in nature. The statement for abelian varieties of dimension n thus implies the statement for abelian varieties of lower dimension.

Step 2: The case $n = 2$ is assumed. We prove next that if the statement holds for abelian varieties of dimension $2k$, $k \geq 1$, then it holds for abelian varieties of dimension $3k$. Write $V = V_{E^{3k}}$. Then $V = V_1 \oplus V_2 \oplus V_3$, where $V_i = V_{E^k}$, for $1 \leq i \leq 3$. Both representations ρ and ρ' of $\mathrm{Spin}(V)$ factor through $SO^+(V)$. Set

$$\begin{aligned} G_1 &:= SO^+(V_1) \times SO^+(V_2 \oplus V_3), \\ G_2 &:= SO^+(V_2) \times SO^+(V_1 \oplus V_3), \\ G_3 &:= SO^+(V_3) \times SO^+(V_1 \oplus V_2). \end{aligned}$$

Let $f : G_1 \times G_2 \times G_3 \to SO^+(V)$ send (g_1, g_2, g_3) to $g_1 g_2 g_3$. The subset of $SO^+(V)$, consisting of elements g for which equality (6.1.10) holds, is a subgroup. Hence, it suffices to show that the Zariski closure in $SO^+(V_{\mathbb{Q}})$ of the subgroup generated by the image of f is the whole of $SO^+(V_{\mathbb{Q}})$. The dimension of $SO^+(V_{\mathbb{Q}})$ is $72k^2 - 6k$ and any closed subgroup of $SO^+(V_{\mathbb{Q}})$ of dimension larger than $(12k-1)(12k-2)/2$ is equal to $SO^+(V_{\mathbb{Q}})$, by [Ob, Theorem B]. Hence, it suffices to prove that the dimension of the Zariski closure of the image of f is $\dim(SO^+(V_{\mathbb{Q}})) - 2$.

[11] Recall that given a group G and a $\mathbb{Z}G$-module M the group cohomology $H^k(G, M)$ is defined to be $\mathrm{Ext}^k_{\mathbb{Z}G}(\mathbb{Z}, M)$. Given a 1-cocycle $\nu : G \to M$, the extention of $\mathbb{Z}$ by M corresponding to ν is the abelian group $\mathbb{Z} \oplus M$, on which $g \in G$ acts via $g(z, m) = (z, g(m) + z\nu_g)$.

We regard each G_i as a subgroup of $SO^+(V_{\mathbb{Q}})$. Assume that $f(g_1,g_2,g_3)=1$, where $g_i\in G_i$. Then g_1g_2 belongs to G_3 and so it maps V_3 to itself. We claim that g_2 maps each V_i to itself, $1\leq i\leq 3$. It suffices to prove that it maps V_3 to itself. Assume otherwise and let $x\in V_3$, such that $g_2(x)=y_1+y_2$, with $y_1\in V_1$ and $y_2\in V_3$ and $y_1\neq 0$. Then $g_1g_2(x)=g_1(y_1)+g_1(y_2)$ and $g_1(y_2)$ belongs to $V_2\oplus V_3$ and $g_1(y_1)$ is a non-zero element of V_1. Hence, $g_1g_2(x)$ does not belong to V_3. A contradiction. We conclude that g_2 maps V_3 to itself as well, and so it maps each V_i to itself. Hence, g_1 maps V_3 to itself, and hence it maps each V_i to itself. The same must thus follow also for g_3. We conclude that the fiber of $f:G_1\times G_2\times G_3\rightarrow SO^+(V_{\mathbb{Q}})$ over 1 has dimension $2[3\dim SO^+(V_{i,\mathbb{Q}})+1]=12k(4k-1)+2$. Now $\dim(G_i)=2k[20k-3]$. Hence, the dimension of the image of f is

$$6k[20k-3]-[12k(4k-1)+2]=72k^2-6k-2=\dim SO^+(V_{\mathbb{Q}})-2.$$

Step 3: We prove next the case $n=4$. We decompose V_{E^4} as the direct sum $V_1\oplus V_2\oplus V_3$, where $V_1=V_2=V_{E^1}$ and $V_3=V_{E^2}$. We set G_i, $1\leq i\leq 3$, and $f:G_1\times G_2\times G_3\rightarrow SO^+(V)$ as in Step 2. Then $\dim(G_1\times G_2\times G_3)=200$, $\dim SO^+(V_{\mathbb{Q}})=120$, and the same argument as in Step 2 shows that the fiber $f^{-1}(1)$ consists of triples (g_1,g_2,g_3), such that each g_i maps each V_j to itself. Hence, the dimension of the fiber $f^{-1}(1)$ is 82. Again we get that the closure of the image of f has codimension 2. Hence, the Zariski closure of the subgroup generated by the image of f is equal to $SO^+(V_{\mathbb{Q}})$.

Step 4: We complete the proof by induction on n. The case $n=2$ is assumed and implies the case $n=1$, by Step 1. The case $n=3$ follows from Step 2 and the case $n=4$ from Step 3. Assume that $n\geq 4$ and the statement holds for E^k, for $k\leq n$. We prove it for E^{n+1}. If n is even, then the statement holds for $E^{3n/2}$, by Step 2 and the induction hypothesis, and so for $n+1$, by Step 1, since $n+1<3n/2$. If n is odd, then $n\geq 5$, the statement holds for $E^{3(n-1)/2}$ by Step 2 and the induction hypothesis, and so for $n+1$, by Step 1, since $n+1\leq 3(n-1)/2$. $\square$

6.2. **Objects in $D^b(X\times\hat{X})$ with $\mathrm{Spin}(V)_P$-invariant Chern classes.** Given a point $x\in X$, let $\tau_x:X\rightarrow X$ be the translation by x, given by $\tau_x(y)=x+y$. Let $a:X\times X\rightarrow X$ be the addition morphism, given by $a(x,y)=x+y$. Let π_{ij} be the projection from $X\times X\times\hat{X}$ onto the product of the i-th and j-th factors. Given a coherent sheaf F over X we get the sheaf

$$\mathcal{F}:=(\pi_{12}^*(a^*(F))\otimes\pi_{13}^*\mathcal{P}^{-1} \tag{6.2.1}$$

over $X\times[X\times\hat{X}]$. The sheaf $\mathcal{F}$ restricts to $X\times\{(x,L)\}$ as $\tau_x^*(F)\otimes L^{-1}$. If F is stable in some sense and corresponds to a point $[F]$ in some fine moduli space $\mathcal{M}$ of stable coherent sheaves over X with a universal sheaf $\mathcal{U}$ over $X\times\mathcal{M}$, then $\mathcal{F}$ is the tensor product of the pullback via π_{23} of some line bundle with the pullback of $\mathcal{U}$ via the morphism $id_X\times\iota_F$, where

$$\iota_F:X\times\hat{X}\rightarrow\mathcal{M} \tag{6.2.2}$$

sends (x,L) to $\tau_x^*(F)\otimes L^{-1}$. The morphism ι_F is associated to the point $[F]$ via the natural action of $X\times\hat{X}$ on $\mathcal{M}$. The following statement relates the construction in [M2, Theorem 1.5 and 13.4] for proving the algebraicity of the Weil classes on abelian fourfolds of Weil-type of discriminant 1, to our strategy in the current paper. In [M2] the right hand side of Equation (6.2.3) was used to construct a coherent sheaf over $X \times \hat{X}$ from a pair of sheaves on X, while in the current paper we use the left hand side.

Lemma 6.2.1. *The isomorphism*

$$\Phi(F_2 \boxtimes F_1^\vee) \cong R\pi_{23,*}(\pi_1^* F_1^\vee \otimes \mathcal{F}_2)[n], \tag{6.2.3}$$

holds for all objects F_i, $i = 1, 2$, in $D^b(X \times X)$.

Proof. Let p_i be the projection from $X \times X \times X \times \hat{X}$ onto the i-th factor.

$$\begin{array}{rcl} (id \times \Psi_{\mathcal{P}^{-1}[n]})(\mu^*(F_2 \boxtimes F_1^\vee))) & \cong & \\ Rp_{34,*}\left\{p_2^*(F_1^\vee) \otimes p_{12}^*(a^*(F_2)) \otimes p_{13}^*(\mathcal{O}_{\Delta_X}) \otimes p_{24}^*(\mathcal{P}^{-1}[n])\right\} & \cong & \\ R\pi_{23,*}\left\{\pi_1^*(F_1^\vee) \otimes \pi_{12}^*(a^*(F_2)) \otimes \pi_{13}^*\mathcal{P}^{-1}\right\}[n] & \cong & R\pi_{23,*}(\pi_1^* F_1^\vee \otimes \mathcal{F}_2)[n]. \end{array}$$

□

Definition 6.2.2. When $ch(F_i)$ belongs to the K-secant P, for $i = 1, 2$, we will refer to the image (6.2.3) of $F_2 \boxtimes F_1^\vee$ via Orlov's equivalence as a *secant*$^{\boxtimes 2}$*-object* in $D^b(X \times \hat{X})$.

Let F_1, F_2 be objects in $D^b(X)$, such that $ch(F_i)$ belongs to the K-secant P, for $i = 1, 2$, such that the 2-form Θ_P in Corollary 3.2.3 is ample. Denote by $H^2(X \times \hat{X}, \mathbb{Q})_P$ the direct sum of all non-trivial irreducible $\mathrm{Spin}(V)_P$-subrepresentations of $H^2(X \times \hat{X}, \mathbb{Q})$. Then $H^2(X \times \hat{X}, \mathbb{Q}) = H^2(X \times \hat{X}, \mathbb{Q})_P + \mathbb{Q}\Theta_P$, by Lemma 2.2.7. Let k be the minimal non-negative integer, such that $ch_k(\Phi(F_2 \boxtimes F_1^\vee)) \neq 0$.

Lemma 6.2.3. *Assume that $k < \dim_{\mathbb{C}}(X)$. There exists a unique class ℓ of type $(1,1)$ in $H^2(X \times \hat{X}, \mathbb{Q})_P$, such that the class $\alpha := \exp(\ell) ch(\Phi(F_2 \boxtimes F_1^\vee))$ is $\mathrm{Spin}(V)_P$-invariant. The class ℓ depends on the secant line P, but not on the choice of F_i, $i = 1, 2$.*

Proof. Set $w_i := ch(F_i)$. The class $w_2 \otimes w_1^\vee = ch(F_2 \boxtimes F_1^\vee)$ is $\mathrm{Spin}(V)_P$-invariant with respect to the diagonal action via $m \times m^\dagger$ on $S_X \otimes S_X$, by Remark 5.2.3. Let β be the class $ch(\Phi(F_2 \boxtimes F_1^\vee))$ in $H^*(X \times \hat{X}, \mathbb{Q})$. Then β is $\mathrm{Spin}(V)_P$-invariant with respect to the ρ' representation. Set $\beta_i := ch_i(\Phi(F_2 \boxtimes F_1^\vee))$. Given $g \in \mathrm{Spin}(V)_P$ there exists a topological complex line-bundle N_g on $X \times \hat{X}$, such that

$$\beta = ch(N_g)\rho_g(\beta), \tag{6.2.4}$$

by Equation (6.1.8). Note that β_k is $\mathrm{Spin}(V)_P$-invariant with respect to ρ, by the minimality of k, and is hence a non-zero scalar multiple of Θ_P^k, by the assumption that $k < \dim_{\mathbb{C}}(X)$. In particular, the homomorphism $\beta_k \cup (\bullet) : H^2(X \times \hat{X}, \mathbb{Q}) \to H^{2k+2}(X \times \hat{X}, \mathbb{Q})$ is injective, as Θ_P is ample.

The coset $\beta_{k+1} + \beta_k \cup H^2(X \times \hat{X}, \mathbb{Q})$ is $\mathrm{Spin}(V)_P$-invariant. Hence, β_{k+1} belongs to the sum of $\beta_k \cup H^2(X \times \hat{X}, \mathbb{Q})$ and the $\mathrm{Spin}(V)_P$-invariant subspace $H^{2k+2}(X \times \hat{X}, \mathbb{Q})^{\mathrm{Spin}(V)_P}$. Thus, there exists a uniques class $\ell \in H^2(X \times \hat{X}, \mathbb{Q})_P$, such that $\gamma := \beta_{k+1} + \beta_k \ell$ is $\mathrm{Spin}(V)_P$-invariant. Given $g \in \mathrm{Spin}(V)_P$, Equation (6.2.4) yields the first equality below. The invariance of β_k and γ yields the second equality.

$$\begin{aligned}\rho_g(\beta_{k+1}) &= -c_1(N_g)\beta_k+\beta_{k+1}=\gamma-[c_1(N_g)+\ell]\beta_k\\ \rho_g(\beta_{k+1}) &= \rho_g(\gamma-\beta_k\ell)=\gamma-\beta_k\rho_g(\ell).\end{aligned}$$

We conclude that

$$c_1(N_g)=\rho_g(\ell)-\ell,$$

for all $g\in \mathrm{Spin}(V)_P$, by the injectivity of the cup product with β_k. In particular, $\ell=0$, if and only if $c_1(N_g)=0$, for all $g\in \mathrm{Spin}(V)_P$. Furthermore, ℓ is determined by P and is independent of the choices of F_i with $ch(F_i)\in P$, $i=1,2$. Indeed, if $c_1(N_g)=\rho_g(\ell')-\ell'$, for all $g\in\mathrm{Spin}(V)_P$, for some $\ell'\in H^2(X\times\hat{X},\mathbb{Q})_P$, then $\ell-\ell'$ is a $\mathrm{Spin}(V)_P$-invariant class in $H^2(X\times\hat{X},\mathbb{Q})_P$, hence $\ell'=\ell$.

Set $\alpha:=\exp(\ell)\beta$. Then $\alpha_{k+1}=\gamma$ and α is $\mathrm{Spin}(V)_P$-invariant, since $\rho_g(\alpha)=\rho_g(\exp(\ell))\rho_g(\beta)=\exp(\rho_g(\ell))ch(N_g^{-1})\beta=\exp(\rho_g(\ell))\exp(\ell-\rho_g(\ell))\beta=\exp(\ell)\beta=\alpha$. □

Remark 6.2.4. Assume $k=0$, so that the rank r of $\Phi(F_2\boxtimes F_1^\vee)$ is non-zero. Set $\beta_1:=c_1(\Phi(F_2\boxtimes F_1^\vee))$. The class

$$\kappa(\Phi(F_2\boxtimes F_1^\vee)):=\exp(-\beta_1/r)ch(\Phi(F_2\boxtimes F_1^\vee))$$

is then $\mathrm{Spin}(V)_P$-invariant, by Lemma 6.2.3. In this case $\ell=(t\Theta_P-\beta_1)/r$, where t is the unique scalar, such that ℓ belongs to $H^2(X\times\hat{X},\mathbb{Q})_P$.

Lemma 6.2.5. *Proposition 6.1.2 holds in case X is an abelian surface.*

Proof. Choose F_1 and F_2 to be ideal sheaves of length n subschemes with Chern character $w_n=(1,0,-n)$, $n\geq 1$. Set $E:=R\pi_{23,*}(\pi_1^*F_1^\vee\otimes\mathcal{F}_2)$. The first three graded summands of the Chern character of E are

$$ch(E)=-2n-nc_1(\mathcal{P})+\left[-\frac{n}{2}c_1(\mathcal{P})^2+n^2\pi_X^*[pt_X]+\pi_{\hat{X}}^*[pt_{\hat{X}}]\right]+\cdots,$$

by the proof of [M2, Prop. 11.2] (the definition of ι_F there is ours, given in (6.2.2), composed with the automorphism of $X\times\hat{X}$ of multiplication by -1 and that automorphism acts as the identity on the even cohomology, so that the result there applies here as well). The first Chern class of the object $E:=R\pi_{23,*}(\pi_1^*F_1^\vee\otimes\mathcal{F}_2)$ in Lemma 6.2.3 is $\beta_1=-nc_1(\mathcal{P})$. On the other hand, the K-secant P can be chosen to be $\mathrm{span}_{\mathbb{Q}}\{w_n,h\}$, where $h\in w_n^\perp$ and $(h,h)_{S^+}<0$. The class ℓ is the projection[12] of $-\frac{c_1(\mathcal{P})}{2}$ to $H^2(X\times\hat{X},\mathbb{Q})_P$, by Remark 6.2.4. We conclude the equality

$$c_1(N_g)=\frac{1}{2}\left[c_1(\mathcal{P})-\rho_g(c_1(\mathcal{P}))\right], \tag{6.2.5}$$

for all $g\in\mathrm{Spin}(V)_P$, for every negative definite rational plane P containing w_n, for some $n\geq 1$. The cocycle identity (6.1.9) implies that Equation (6.2.5) holds for every element g of the subgroup Γ of $\mathrm{Spin}(V)$ generated by the union

$$\cup\{\mathrm{Spin}(V)_P\ :\ P \text{ is a negative definite rational plane containing } w_n, \text{ for some } n\geq 1\}.$$

[12] The $\mathrm{Spin}(V)_P$-invariant class Θ_P clearly depends on P (under the identification of $\wedge^2V_{\mathbb{Q}}$ and $\wedge^2S^+_{\mathbb{Q}}$ the class Θ_P is a scalar multiple of $w_n\wedge h$, by [M2, (12.5)]). So β_1 need not be a scalar multiple of Θ_P.

Equation (6.1.10) holds for all $g \in \Gamma$. Passing to $\mathbb{Q}$ coefficients, Equation (6.1.10) holds for a Zariski closed subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})$ and in particular for g in the Zariski closure of Γ. The latter is the whole of $\mathrm{Spin}(V_{\mathbb{Q}})$ (see for example [Ve, Th. 2.1]). $\square$

Let $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ be the group appearing in Lemma 2.2.7.

Lemma 6.2.6. *Let E be an object of $D^b(X \times \hat{X})$ of non-zero rank r and let G be a subgroup of $\mathrm{Spin}(V)$.*

1. *The class $ch(E)$ is G-ρ'-invariant, if and only if both $\kappa(E)$ and $c_1(E) - \frac{r}{2}c_1(\mathcal{P})$ are G-ρ-invariant.*
2. *Assume that $ch(E)$ is $\mathrm{Spin}(V)_P$-ρ'-invariant. Then $ch(E)$ is $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho'$-invariant, if and only if $\kappa(E)$ is $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho$-invariant.*

Proof. (1) Proposition 6.1.2 implies that for all $g \in \mathrm{Spin}(V)$, $ch(E)$ is ρ'_g-invariant, if and only if $ch(E) \cup \exp(-\frac{1}{2}c_1(\mathcal{P})) = r + [c_1(E) - \frac{r}{2}c_1(\mathcal{P})] + \cdots$ is ρ_g-invariant. The latter is ρ_g-invariant, if and only if $c_1(E) - \frac{r}{2}c_1(\mathcal{P})$ and $\kappa(E)$ are both ρ_g-invariant.

(2) Lemma 2.2.7 implies that the $\mathrm{Spin}(V)_P$-ρ-invariant subspace $H^2(X \times \hat{X}, \mathbb{Z})^{\mathrm{Spin}(V)_P}$ is also $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho$-invariant. Hence, the statement follows from part (1). $\square$

6.3. **Orlov's equivalence induces Chevalley's isomorphism $S \otimes S \cong \wedge^* V$.** Let $\beta := \{e_1, \ldots, e_{2n}\}$ be a basis of $H^1(X, \mathbb{Z})$ and let $\{f_1, \ldots, f_{2n}\}$ be the dual basis of $H^1(\hat{X}, \mathbb{Z})$. Define $\mu : X \times X \to X \times X$ by $\mu(x, y) = (x + y, y)$. Then

$$\begin{aligned} \mu^*(\pi_1^*(e_i)) &= \pi_1^*(e_i) + \pi_2^*(e_i), \\ \mu^*(\pi_2^*(e_i)) &= \pi_2^*(e_i). \end{aligned}$$

Let $\nu : H^*(X \times X, \mathbb{Z}) \to H^*(\hat{X} \times X, \mathbb{Z})$ be the isomorphism induced by the equivalence $(\Psi_{\mathcal{P}^{-1}[n]} \otimes 1) \circ \mu^* : D^b(X \times X) \to D^b(\hat{X} \times X)$. Let $\tilde{\varphi} : H^*(X \times X, \mathbb{Z}) \to H^*(\hat{X} \times X, \mathbb{Z})$ be the isomorphism (2.3.2).

Lemma 6.3.1. *The equality $\nu \circ (id \otimes \tau) = \tilde{\varphi}$ holds.*

Proof. Given a subset $K := \{i_1, i_2, \ldots, i_k\}$ of $[1, 2n] := \{1, 2, \ldots, 2n\}$, ordered by the induced ordering of $[1, 2n]$ so that $i_t < i_{t+1}$, set $e_K := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ and $f_K := f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_{i_k}$. Let $L = \{j_1, \ldots, j_\ell\}$ be another subset, with $j_t < j_{t+1}$, and define e_L similarly. Given a subset $I \subset K$, we denote by I' its complement in K and by I^c its complement in $[1, 2n]$. Define $\epsilon_{K,L} \in \{-1, 0, 1\}$ by the equality $e_K \wedge e_L = \epsilon_{K,L} e_{K \cup L}$. So $\epsilon_{K,L} = 0$, if $K \cap L \neq \emptyset$. Note that $\epsilon_{K,K^c} = (-1)^{\sum(K) - k(k+1)/2}$, where $\sum(K) := \sum_{t=1}^k i_t$. Furthermore, $\epsilon_{K^c,K} = (-1)^k \epsilon_{K,K^c} = (-1)^{\sum(K) - k(k-1)/2}$. We have $e_K \wedge \epsilon_{K,K^c} e_{K^c} = (\epsilon_{K,K^c})^2 e_{K \cup K^c} = e_1 \wedge \cdots \wedge e_{2n}$. So Poincaré duality PD_X sends e_K to $\int_X e_K \wedge (\bullet) = \epsilon_{K,K^c} f_{K^c}$. In this notation we have

$$\begin{aligned} \mu^*(\pi_1^* e_K \wedge \pi_2^* e_L) &= [\pi_1^* e_{i_1} + \pi_2^* e_{i_1}] \wedge \cdots \wedge [\pi_1^* e_{i_k} + \pi_2^* e_{i_k}] \wedge \pi_2^* e_L \\ &= \sum_{I \subset K} \epsilon_{I,I'} \pi_1^* e_I \wedge \pi_2^*(e_{I'} \wedge e_L) \\ &= \sum_{I \subset K} \epsilon_{I,I'} \epsilon_{I',L} \pi_1^* e_I \wedge \pi_2^*(e_{I' \cup L}). \end{aligned}$$

The cohomological action of the functor $\Psi_{\mathcal{P}^{-1}[n]}: D^b(X)\rightarrow D^b(\hat{X})$ is given by

$$\Psi_{\mathcal{P}^{-1}[n]}(e_K)=\sigma_K PD(e_K)=\sigma_K\epsilon_{K,K^c}f_{K^c},$$

where[13] $\sigma_K=(-1)^{k(k+3)/2}$. We get

$$\nu(\pi_1^*e_K\wedge\pi_2^*e_L)=\sum_{I\subset K}\epsilon_{I,I'}\epsilon_{I',L}\sigma_I\epsilon_{I,I^c}\pi^*_{\hat{X}}f_{I^c}\wedge\pi^*_X(e_{I'\cup L}).$$

Interchanging I and I', with respect to which the sum is symmetric, we get

$$\begin{aligned}\nu(\pi_1^*e_K\wedge\pi_2^*e_L) &= \sum_{I\subset K}\epsilon_{I',I}\epsilon_{I,L}\sigma_{I'}\epsilon_{I',(I')^c}\pi^*_{\hat{X}}f_{(I')^c}\wedge\pi^*_X(e_{I\cup L})\\ &= \sum_{I\subset K}\epsilon_{I',I}\epsilon_{I,L}(-1)^{\sum(I')-|I'|}\pi^*_{\hat{X}}f_{(I')^c}\wedge\pi^*_X(e_{I\cup L}).\end{aligned}\tag{6.3.1}$$

Consider next the isomorphism $\tilde{\varphi}: S\otimes_{\mathbb{Z}}S\rightarrow\wedge^*V$ of section 2.3. We chose the bilinear pairing $B_0(\bullet,\bullet)$ on V with respect to which $H^1(X,\mathbb{Z})$ and $H^1(\hat{X},\mathbb{Z})$ are isotropic, and satisfing $B_0(e_i,f_j)=\delta_{i,j}$ and $B_0(f_i,e_j)=0$, for all $1\leq i,j\leq 2n$. Let $\psi': C(V)\rightarrow \mathrm{End}(\wedge^*V)$ send v to $L_v+\delta_{B_0(v,\bullet)}$, where δ_x, $x\in V^*$, is contraction with x. We denote $\delta_{B_0(e_i,\bullet)}$ by δ_{e_i} for short using the equality $B_0(e_i,\bullet)=(e_i,\bullet)_V$ and the identifiation of V with V^* via $(\bullet,\bullet)_V$. So $\psi'(e_i)=L_{e_i}+\delta_{e_i}$ and $\psi'(f_i)=L_{f_i}$. We set $\delta_K:=\delta_{e_{i_1}}\delta_{e_{i_2}}\cdots\delta_{e_{i_k}}$. Let $\psi: C(V)\rightarrow\wedge^*V$ be ψ' composed with evaluation at the unit $\mathbb{1}$. Set $\tilde{\varphi}(s\otimes t)=\psi(s[pt_{\hat{X}}]\tau(t))$. Then

$$\begin{aligned}\tilde{\varphi}(e_K\otimes e_L) &= [L_{e_{i_1}}+\delta_{e_{i_1}}]\circ\cdots\circ[L_{e_{i_k}}+\delta_{e_{i_k}}]f_1\wedge\cdots\wedge f_{2n}\wedge\tau(e_L)\\ &= \sum_{I\subset K}\epsilon_{I,I'}e_I\wedge\delta_{I'}(f_1\wedge\cdots\wedge f_{2n})\wedge(-1)^{\ell(\ell-1)/2}e_L\\ &= \sum_{I\subset K}\epsilon_{I,I'}(-1)^{\ell(\ell-1)/2}(-1)^{|I|(2n-|I'|)}\delta_{I'}(f_1\wedge\cdots\wedge f_{2n})\wedge e_I\wedge e_L\\ &= \sum_{I\subset K}\epsilon_{I,I'}(-1)^{\ell(\ell-1)/2}(-1)^{|I||I'|}(-1)^{\sum(I')-|I'|}\epsilon_{I,L}f_{(I')^c}\wedge e_{I\cup L}\\ &= \sum_{I\subset K}\epsilon_{I',I}(-1)^{\ell(\ell-1)/2}(-1)^{\sum(I')-|I'|}\epsilon_{I,L}f_{(I')^c}\wedge e_{I\cup L},\end{aligned}$$

where in the fourth equality we used the equality $\delta_{I'}(f_1\wedge\cdots\wedge f_{2n})=(-1)^{\sum(I')-|I'|}f_{(I')^c}$. Comparing with (6.3.1) we get

$$\nu(\pi_1^*(e_K)\wedge\pi_2^*\tau(e_L))=\tilde{\varphi}(e_K\otimes e_L).\tag{6.3.2}$$

This completes the proof of Lemma 6.3.1. □

[13] $\phi_{\mathcal{P}}: H^k(\hat{X},\mathbb{Z})\rightarrow H^{2n-k}(X,\mathbb{Z})$ is equal to $(-1)^{k(k+1)/2+n}PD_k$, where $PD_k: H^k(\hat{X},\mathbb{Z})\rightarrow H^{2n-k}(X,\mathbb{Z})$ is the Poincaré duality isomorphism [H2, Lemma 9.23]. Furthermore, the composition

$$H^k(\hat{X},\mathbb{Z})\stackrel{\phi_{\mathcal{P}}}{\rightarrow}H^{2n-k}(X,\mathbb{Z})\stackrel{\phi_{\mathcal{P}}}{\rightarrow}H^k(\hat{X},\mathbb{Z})$$

is $(-1)^{k+n}$ [H2, Cor. 9.24]. Now, $\Psi_{\mathcal{P}^{-1}}[n]$ is the inverse of $\Phi_{\mathcal{P}}$ and so the the cohomological action of $\Psi_{\mathcal{P}^{-1}}[n]$ restricts to $H^k(X,\mathbb{Z})$ as $(-1)^{k+n}\phi_{\mathcal{P}}=(-1)^{k(k+3)/2}PD_k$.

Proof of Lemma 6.1.1. We have

$$\phi := (id\otimes \psi_{\mathcal{P}^{-1}[n]})\circ \mu^* = (\phi_{\mathcal{P}}\otimes \psi_{\mathcal{P}^{-1}[n]})\circ \nu = (\phi_{\mathcal{P}}\otimes \psi_{\mathcal{P}^{-1}[n]})\circ \tilde{\varphi}\circ (id\otimes \tau),$$

where the first equality is the definition of ϕ, the second is the definition of ν, and the third is Lemma 6.3.1. $\square$

Lemma 6.3.2. $(\phi_{\mathcal{P}}\otimes\psi_{\mathcal{P}^{-1}}) : H^d(\hat{X}\times X) \to H^{4n-d}(X\times\hat{X})$ *is equal to* $(-1)^{d(d+1)/2}PD_{\hat{X}\times X}$.

Proof. Keep the notation of the proof of Lemma 6.3.1. In particular, $f_L = f_{j_1}\wedge\cdots\wedge f_{j_\ell}$ and $e_K = e_{i_1}\wedge\cdots\wedge e_{i_k}$ are classes of degrees ℓ and k respectively. We have the equality $\phi_{\mathcal{P}}^{-1} = \psi_{\mathcal{P}^{-1}}$. By definition, $(PD_{\hat{X}\times X}(f_L\wedge e_K),\bullet) = \int_{\hat{X}\times X}(f_L\wedge e_K)\wedge\bullet$.

$$\begin{array}{rcl}(\phi_{\mathcal{P}}\otimes\phi_{\mathcal{P}}^{-1})(f_L\wedge e_K) & = & (-1)^{\ell(\ell+1)/2+n}(-1)^{(2n-k)(2n-k+1)/2+n}PD(f_L)\wedge PD^{-1}(e_K)\\ & = & (-1)^{[k(k+1)+\ell(\ell+1)]/2}(-1)^k\epsilon_{L,L^c}\epsilon_{K^c,K}e_{L^c}\wedge f_{K^c}\\ & = & (-1)^{[k(k+1)+\ell(\ell+1)]/2}\epsilon_{L,L^c}\epsilon_{K,K^c}e_{L^c}\wedge f_{K^c}\end{array}$$

So

$$\begin{array}{rcl}((\phi_{\mathcal{P}}\otimes\phi_{\mathcal{P}}^{-1})(f_L\wedge e_K), f_{L^c}\wedge e_{K^c}) & = & (-1)^{[k(k+1)+\ell(\ell+1)]/2}\epsilon_{L,L^c}\epsilon_{K,K^c}\\ & = & (-1)^{(k+\ell)(k+\ell+1)/2}(-1)^{kl}\epsilon_{L,L^c}\epsilon_{K,K^c}.\end{array}$$

On the other hand, $\int_{\hat{X}\times X}(f_L\wedge e_K)\wedge(f_{L^c}\wedge e_{K^c}) = (-1)^{kl}\int_{\hat{X}\times X} f_L\wedge f_{L^c}\wedge e_K\wedge e_{K^c} = (-1)^{kl}\epsilon_{L,L^c}\epsilon_{K,K^c}$. $\square$

6.4. **Hodge-Weil classes on $X\times\hat{X}$ from tensor squares of even pure spinors.** Let $P\subset S^+_{\mathbb{Q}}$ be a rational plane satisfying Assumption 2.4.1. We keep the notation of Section 2.2. In particular $\ell_i\in\mathbb{P}(P)$, $i=1,2$, are the two complex conjugate pure spinors, $\tilde{\ell}_i\subset P_K$ is the 1-dimensional subspace corresponding to ℓ_i, and $W_i\subset V_K$ is the corresponding maximal isotropic subspace. The subspace $P\otimes P$ of $S^+_{\mathbb{Q}}\otimes S^+_{\mathbb{Q}}$ is a trivial $\mathrm{Spin}(V_{\mathbb{Q}})_P$ sub-representation, but it decomposes as a direct sum of one-dimensional $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-representations, the two non-trivial distinct complex conjugate characters $\tilde{\ell}_1^{\otimes 2}$, $\tilde{\ell}_2^{\otimes 2}$, and the two trivial characters $\tilde{\ell}_1\wedge\tilde{\ell}_2$ in $\wedge^2 S^+_{\mathbb{Q}}$ and $\tilde{\ell}_1\tilde{\ell}_2$ in $\mathrm{Sym}^2(S^+_{\mathbb{Q}})$. Note that the 2-dimensional subspace

$$HW_{P_K} := \tilde{\ell}_1^{\otimes 2}\oplus\tilde{\ell}_2^{\otimes 2}$$

of $S^+_K\otimes_K S^+_K$ is defined over $\mathbb{Q}$. Denote by HW_P the corresponding 2-dimensional subspace of $S^+_{\mathbb{Q}}\otimes S^+_{\mathbb{Q}}$.

Let $\rho,\rho' : \mathrm{Spin}(V)\to GL(\wedge^*V)\cong GL(H^*(X\times\hat{X},\mathbb{Z}))$ be the two representations given in (6.1.6) and (6.1.7). Both factor though the image $SO^+(V)$ of $\mathrm{Spin}(V)$. Recall that ρ is the graded extension of the natural homomorphism $\rho : \mathrm{Spin}(V)\to SO^+(V)$. Given $g\in\mathrm{Spin}(V)$, the automorphism $\rho'(g)$ preserves the decreasing filtration

$$F^kH^*(X\times\hat{X},\mathbb{Z}) := \oplus_{i\geq k}H^i(X\times\hat{X},\mathbb{Z}),$$

$0\leq k\leq 4n$. The induced action of $\rho'(g)$ on the graded summands agrees with that of $\rho(g)$, by Proposition 6.1.2.

Let $U\subset H^*(X\times\hat{X},K)$ be an irreducible representation of the restriction of the ρ' action of $\mathrm{Spin}(V_K)$ to a subgroup $G\subset\mathrm{Spin}(V_K)$. Let $k(U)$ be the maximal integer, such that U is contained in $F^kH^*(X\times\hat{X},K)$. We call $k(U)$ the *weight* of U. The intersection of U with the subspace $F^{k(U)+1}H^*(X\times\hat{X},K)$ vanishes, as it is a proper subrepresentation of the irreducible representation U. Hence, U projects injectively and G-equivariantly into an irreducible representation $\hat{U}\subset H^{k(U)}(X\times\hat{X},K)$ of G via the ρ action of $\mathrm{Spin}(V_K)$. If U is reducible, but each irreducible G-subrepresentation of U has the same weight k_0, we set $k(U)=k_0$ and let $\hat{U}$ be the projection of U in $H^{k(U)}(X\times\hat{X},K)$. Again we call $k(U)$ the *weight* of U. If, furthermore, U decomposes as a direct sum of pairwise distinct irreducible subrepresentations of the same weight $k(U)$, then again the projection homomorphism $U\to\hat{U}$ is an isomorphism of G representations. Set $\phi':=\phi\circ(id\otimes\tau):S\otimes S\to\wedge^*V$, where ϕ given in (6.1.3). By definition (6.1.4), $\rho'_g=\phi'(m_g\otimes m_g)\phi'^{-1}$.

Proposition 6.4.1. (1) *The isomorphism ϕ' maps $HW_P:=\tilde{\ell}_1^{\otimes 2}\oplus\tilde{\ell}_2^{\otimes 2}$ into a weight $2n$ $\mathrm{Spin}(V)_P$-subrepresentation of $H^*(X\times\hat{X},\mathbb{Q})$ via ρ' and the projection $\hat{HW}_P$ of $\phi'(HW_P)$ is the rational subspace of $H^{2n}(X\times\hat{X},\mathbb{Q})$ corresponding to the subspace $\wedge^{2n}W_1\oplus\wedge^{2n}W_2$ of $H^{2n}(X\times\hat{X},K)$.*

(2) *The weights of $\phi'(\ell_1\wedge\ell_2)$ and $\phi'(\ell_1\cdot\ell_2)$, as $\mathrm{Spin}(V)_P$-subrepresentations via ρ', are as follows. If n is even, then the weight of $\phi'(\ell_1\wedge\ell_2)$ is 2 and the weight of $\phi'(\ell_1\cdot\ell_2)$ is 0. If n is odd, then the weight of $\phi'(\ell_1\cdot\ell_2)$ is 2 and the weight of $\phi'(\ell_1\wedge\ell_2)$ is 0.*

Proof. (1) The homomorphism ϕ' is defined over $\mathbb{Q}$. Thus, the two distinct subrepresentations $\phi'(\tilde{\ell}_i^2)$, $i=1,2$, are complex conjugates and hence have the same weight $k:=k(\tilde{\ell}_i^2)$. The weight is even, as ϕ' is $\mathbb{Z}/2\mathbb{Z}$ graded and $P\otimes P$ is contained in $H^{even}(X\times X,\mathbb{Q})$. It follows that $\hat{HW}_{P_K}:=\widehat{\phi'(\tilde{\ell}_1^2)}\oplus\widehat{\phi'(\tilde{\ell}_1^2)}$ is a 2-dimensional subrepresentation of $H^k(X\times\hat{X},K)$, which is defined over $\mathbb{Q}$.

Lemma 2.2.7 shows that the $\mathrm{Spin}(V)_P$-invariant subspace $H^{2i}(X\times\hat{X},\mathbb{Q})^{\mathrm{Spin}(V)_P}$ of $H^{2i}(X\times\hat{X},\mathbb{Q})$ is one-dimensional and it is a trivial $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ character, for $i\neq n$ and $0\leq i\leq 4n$. The space $H^{2n}(X\times\hat{X},\mathbb{Q})^{\mathrm{Spin}(V)_P}$ is three-dimensional and it is the direct sum of the characters $\det_1$, $\det_2$ and the trivial character of $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$. Hence, the weight of $\phi'(\tilde{\ell}_1^2)\oplus\phi'(\tilde{\ell}_1^2)$ is $2n$. We have already seen in the proof of Lemma 2.2.6 that $\tilde{\ell}_i^2$ is the character $\wedge^{2n}W_i\cong\det_i$ of $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$. It follows that $\widehat{\phi'(\tilde{\ell}_i^2)}$ is equal to $\wedge^{2n}W_i$.

(2) The group $\mathrm{Spin}(V_K)$ acts transitively on the set of ordered pairs of complementary maximally isotropic subspaces of V_K, by [Ch, Sec. 3.3, Lemma 1]. The weight is invariant under the $\mathrm{Spin}(V_K)$-action. Hence, it suffices to calculate the weights of the one dimensional representations spanned by of $\phi'(1\wedge[pt_X])$ and $\phi'(1\cdot[pt_X])$. We related the isomorphism $\tilde{\varphi}$ given in (2.3.2) to the isomorphism ϕ given in (6.1.3) in Lemma 6.1.1. We have $\phi'=(\phi_{\mathcal{P}}\otimes\psi_{\mathcal{P}^{-1}[n]})\circ\tilde{\varphi}$. The image of $([pt_X]\otimes 1)-(-1)^n(1\otimes[pt_X])$ via $\tilde{\varphi}$ is contained in $F^{4n-2}(\wedge^*V)$ and not in $F^{4n-3}(\wedge^*V)$, by Lemma 2.3.2(1). Now $(\phi_{\mathcal{P}}\otimes\psi_{\mathcal{P}^{-1}[n]})$ sends $F^k(\wedge^*V)$ to $F_{4n-k}(\wedge^*V)$, by Lemma 6.3.2, and so the weight is 2. Similarly, the weight of the other line is zero, by Lemma 2.3.2(2), via the same argument. □

7. Semiregular twisted sheaves

In Section 7.1 we recall the definition of the semi-regularity map of a coherent sheaf F in terms of its Atiyah class at_F. A sheaf is semiregular, when its semi-regularity map is injective. In Section 7.2 we extend the definition of the semi-regularity map for projective bundles. In Section 7.3 we extend the definition for coherent sheaves twisted by a Čech cocycle with coefficients in the local system μ_r of r-th roots of unity. We state the analogue of the Buchweitz-Flenner Semiregularity Theorem for twisted sheaves as Conjecture 7.3.9. It states that a semiregular twisted sheaf on a special fiber of a family deforms to a twisted sheaf over an open neighborhood of the fiber, provided its Chern character remains of Hodge type. An analogue of the Semiregularity Theorem for twisted sheaves follows from a very general theorem of Pridham [Pr, Remark 2.6]. The author's ignorance prevents him from comparing the two statements, though Conjecture 7.3.9 should follow from Pridham's result. In Section 7.4 we reduce Conjecture 7.3.9 to the Semiregularity Theorem for untwisted sheaves in the case of families of abelian varieties (and more generally, whenever the Brauer class of the twisted sheaf on the special fiber is the restriction of that of a projective bundle over the family).

7.1. **Semiregular coherent sheaves.** Let E be a coherent sheaf on a d-dimensional complex manifold M. Denote the Atiyah class of E by $at_E \in \mathrm{Ext}^1(E, E \otimes \Omega^1_M)$. The q-th component σ_q of the semiregularity map

$$\sigma := (\sigma_0, \ldots, \sigma_{d-2}) : \mathrm{Ext}^2(E,E) \to \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M) \tag{7.1.1}$$

is the composition

$$\mathrm{Ext}^2(E,E) \stackrel{(at_E)^q/q!}{\longrightarrow} \mathrm{Ext}^{q+2}(E, E\otimes \Omega^q_M) \stackrel{Tr}{\longrightarrow} H^{q+2}(M, \Omega^q_M).$$

The sheaf E is said to be *semiregular*, if σ is injective. We have the commutative diagram

$$\begin{tikzcd} H^1(M,TM) \arrow[rr, "at_E"] \arrow[dr, "\rfloor ch(E)"'] & & \mathrm{Ext}^2(E,E) \arrow[dl, "\sigma"] \\ & \prod_{q=0}^{d-2} H^{q+2}(M,\Omega^q_M) & \end{tikzcd} \tag{7.1.2}$$

by [BF1, Cor. 4.3].[14]

7.2. **Semiregular projective bundles.** Let B be a projective $\mathbb{P}^{r-1}$-bundle over M. Denote by $\mathcal{A}$ the Azumaya algebra of B. Then $\mathcal{A}$ is a coherent sheaf of associative algebras with a unit over M, locally isomorphic to the sheaf of endomorphisms of a locally free sheaf. If $B = \mathbb{P}(E)$, for a locally free sheaf E over M, then $\mathcal{A}$ is naturally isomorphic to $\mathcal{E}nd(E)$. Let $\mathcal{A}_0$ be the kernel of the trace homomorphism $tr : \mathcal{A} \to \mathcal{O}_M$.

[14]We follow the convention of [T] and [HL] for the sign of the Atiyah class, so that $ch(E) = Tr(at_E)$, while in [BF1] $ch(E) = Tr(-at_E)$. So for a vector bundle we choose the Atiyah class to be the extension class of the first jet bundle, rather than the extension class of the bundle of first order differential operators with scalar symbol. See [At, Theorem 5].

We have the direct sum decomposition $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{O}_M$, where the second direct summand is generated by the unit. The sheaf $\mathcal{A}_0$ is isomorphic to the adjoint Lie algebra bundle of the principal $PGL(r)$-bundle $Pr(B)$ associated to B. Denote by Q the Atiyah bundle of $Pr(B)$ [At]. It fits in the short exact sequence

$$0 \to \mathcal{A}_0 \to Q \to TM \to 0.$$

Let $-at_B \in H^1(M, \mathcal{A}_0 \otimes \Omega^1_M)$ be the extension class of the above sequence. If $B = \mathbb{P}(E)$, then at_B is the traceless direct summand of at_E, by [At, page 189 property (1)]. Consequently,

$$at_B = at_E - \frac{c_1(E)}{r} \cdot id_E \tag{7.2.1}$$

(keeping the convention of footnote 14).

We view the class at_B as a class in $H^1(M, \mathcal{A} \otimes \Omega^1_M)$ via the inclusion of $\mathcal{A}_0$ as a direct summand in $\mathcal{A}$. We get the exponential Atiyah class $\exp(at_B)$ with graded summands $at_k(B)$ in $H^k(M, \mathcal{A} \otimes \Omega^k_M)$. Note that $at_0(B)$ is r times the unit section and $at_1(B) = at_B$. Denote by $\kappa(B)$ the trace of $\exp(at_B)$. If $B = \mathbb{P}(E)$, then

$$\kappa(B) = ch(E) \exp(-c_1(E)/r),$$

by equation (7.2.1). We get the commutative diagram

$$\begin{tikzcd} H^1(M, TM) \arrow[rr, "at_B"] \arrow[dr, "\rfloor\kappa(B)"'] & & H^2(M, \mathcal{A}_0) \arrow[dl, "\sigma_B"] \\ & \prod_{q=1}^{d-2} H^{q+2}(M, \Omega^q_M), & \end{tikzcd} \tag{7.2.2}$$

where the *semiregularity map* σ_B is defined as in equation (7.1.1) with at_E replaced by at_B. The commutativity of Diagram (7.2.2) is proved by the same argument as in [BF1, Cor. 4.3] establishing the commutativity of Diagram (7.1.2). See Remark 9.3.10 for another proof.

7.3. **Semiregular μ_r-twisted sheaves.** We refer to [Ca1] for basic facts about twisted sheaves. Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open covering of M. Let $\mathcal{B}$ be a coherent sheaf of non-zero rank r over M twisted by a Čech cocycle $\tilde{\theta}$ in $\mathcal{Z}^2(\mathcal{U}, \mu_r)$, where μ_r is the local system of r-th roots of unity. The $\tilde{\theta}$-twisted sheaf $\mathcal{B}$ has a well defined untwisted determinant line-bundle $\wedge^r \mathcal{B}$, since $(\tilde{\theta}_{ijk})^r = 1$.

Example 7.3.1. Every projective bundle B on M admits a lift to such a $\tilde{\theta}$-twisted locally free sheaf $\mathcal{B}$ with trivial determinant line bundle, where $\tilde{\theta}$ is a cocycle representing the characteristic class $\theta \in H^2(M, \mu_r)$ of the bundle. The characteristic class θ is the image of the class of the bundle in $H^1(M, PGL(r, \mathcal{O}_M))$ via the connecting homomorphism of the short exact sequence

$$1 \to \mu_r \to SL(r, \mathcal{O}_M) \to PGL(r, \mathcal{O}_M) \to 1. \tag{7.3.1}$$

Note that the group of line bundles of order r in $\mathrm{Pic}^0(M)$ acts transitively on the set of isomorphism classes of choices of lifts $\mathcal{B}$. Note that if $\mathcal{B}$ happens to be untwisted, so that B admits a lift to a vector bundle $\mathcal{B}$ with trivial determinant, then the Atiyah class of the principal $SL(r)$-bundle associated to $\mathcal{B}$ and the Atiyah class of the projective bundle B are equal, by [At, page 189 property (1)].

Remark 7.3.2. Keep the notation of Example 7.3.1. If the characteristic class θ of the projective bundle B has order ρ in $H^2(M,\mu_r)$, then ρ divides r. Set $k = r/\rho$. The short exact sequence $0 \to \mu_\rho \to \mu_r \stackrel{(\bullet)^\rho}{\to} \mu_k \to 0$ yields exactness of

$$H^2(M,\mu_\rho) \to H^2(M,\mu_r) \stackrel{(\bullet)^\rho}{\to} H^2(M,\mu_k).$$

Hence, θ is the image of a class $\theta' \in H^2(M,\mu_\rho)$. Let $\tilde{\theta}'$ be a Čech 2-cocycle with coefficients in μ_ρ, which is cohomologous in $Z^2(\mathcal{U},\mu_r)$ to the cocycle $\tilde{\theta}$ in Example 7.3.1, $\tilde{\theta}' = \tilde{\theta}\delta(\alpha)$, for a Čech 1-co-chain α with coefficients in μ_r. Multiplying the gluing transformations of $\mathcal{B}$ by the co-chain α we get the $\tilde{\theta}'$-twisted sheaf $\mathcal{B}'$, still with trivial determinant, with $\mathbb{P}(\mathcal{B}') \cong \mathbb{P}(\mathcal{B}) \cong B$. We can thus choose the Čech 2-cocycle $\tilde{\theta}$ in Example 7.3.1 to have coefficients in μ_ρ.

Construction 7.3.3. Let $\mathcal{E}$ be a coherent sheaf of rank $r > 0$ over M. Let $\mathcal{U} := \{U_i\}_{i\in I}$ be an open covering, in the analytic topology, with each U_i biholomorphic to a polydisc and with simply connected finite intersections. Denote by $\mathcal{E}_i$ the restriction of $\mathcal{E}$ to U_i and let $\psi_i : \wedge^r\mathcal{E}_i \stackrel{\cong}{\to} \mathcal{O}_{U_i}$ be a trivialization of the determinant line bundle. The line bundle $\wedge^r\mathcal{E}$ is represented by the Čech cocycle $\eta_{ij} := (\psi_i)_{|U_{ij}} \circ (\psi_j)^{-1}_{|U_{ij}}$. Choose an r-th root $\tilde{\eta}_{ij}$ of the invertible holomorphic function η_{ij}. Let $\varphi_{ij} : (\mathcal{E}_j)_{|U_{ij}} = \mathcal{E}_{|U_{ij}} \to \mathcal{E}_{|U_{ij}} = (\mathcal{E}_i)_{|U_{ij}}$ be multiplication by $\tilde{\eta}_{ij}^{-1}$. Then $\theta_{ijk} := (\tilde{\eta}_{ij}\tilde{\eta}_{jk}\tilde{\eta}_{ki})^{-1}$ is a Čech cocycle θ in $\mathcal{Z}^2(\mathcal{U},\mu_r)$ and $\mathcal{B} := (\{\mathcal{E}_i\},\{\varphi_{ij}\})$ is a θ-twisted sheaf. The line bundle $\wedge^r\mathcal{B}$ is trivial, as its transition functions are $\tilde{\eta}_{ij}^{-r}\psi_i \circ \psi_j^{-1} = (\psi_j \circ \psi_i^{-1}) \circ \psi_i \circ \psi_j^{-1} = 1$. Note that the 2-cocycle θ represents the image of the class of $\det(\mathcal{E})^{-1}$ under the connecting homomorphism $H^1(M,\mathcal{O}_M^\times) \to H^2(M,\mu_r)$ associated to the short exact sequence

$$0 \to \mu_r \to \mathcal{O}_M^\times \stackrel{(\bullet)^r}{\to} \mathcal{O}_M^\times \to 0.$$

If the class $[\theta] \in H^2(X,\mu_r)$ has order ρ, then ρ divides r and we may assume that the cocycle θ has coefficients in the local system $\mu_\rho \subset \mu_r$, by the construction in Remark 7.3.2. The above construction goes through more generally for a twisted sheaf $\mathcal{E} = (\{\mathcal{E}_i\},\phi_{ij})$, by a cocycle $\theta' \in Z^2(\mathcal{U},\mu_r)$, in which case the resulting sheaf $\mathcal{B} := (\{\mathcal{E}_i\},\phi_{ij}\tilde{\eta}_{ij}^{-1})$ with a trivial determinant line bundle is $\theta'\theta$-twisted.

Remark 7.3.4. Note that for the twisted sheaf $\mathcal{B}$ in Construction 7.3.3 we have the natural isomorphism $\mathrm{Ext}^1(\mathcal{B},\mathcal{B}\otimes\Omega_M) \cong \mathrm{Ext}^1(\mathcal{E},\mathcal{E}\otimes\Omega_M)$, since $R\mathcal{H}om(\mathcal{B},\mathcal{B})$ and $R\mathcal{H}om(\mathcal{E},\mathcal{E})$ are naturally isomorphic.

Let M be a smooth variety, $\mathcal{U} := \{U_i\}_{i\in I}$ an open covering, and $\theta := (\theta_{ijk})_{i,j,k\in I}$ a Čech 2-cocycle with coefficients in the local system μ_r. Let $\Delta_M \subset M\times M$ be the diagonal and let $\mathcal{M} \subset M \times M$ the first order infinitesimal neighborhood of Δ_M, i.e., $\mathcal{M}$ is the subscheme of $M\times M$ with ideal sheaf $\mathcal{I}^2_{\Delta_M}$. Let $\pi_i : \mathcal{M} \to M$, $i = 1,2$, be the two projections and let $\delta : M \to \mathcal{M}$ be the inclusion. We have the short exact sequence

$$0 \to \delta_*\Omega_M \to \mathcal{O}_\mathcal{M} \stackrel{\delta^*}{\to} \delta_*\mathcal{O}_M \to 0. \tag{7.3.2}$$

Its extension class is called the *universal Atiyah class* and is a morphism

$$at : \delta_*\mathcal{O}_M \to \delta_*\mathcal{O}_{\Omega_M}[1] \tag{7.3.3}$$

in $D^b(M \times M)$, which we regard as a natural transformation $at : id \to \Omega_M[1] \otimes (\bullet)$ of endofunctors of $D^b(M)$. Note that the open coverings $\{\pi_1^{-1}(U_i)\}_{i\in I}$ and $\{\pi_2^{-1}(U_i)\}_{i\in I}$ of $\mathcal{M}$ coincide. The 2-cocycles $(\pi_1^*\theta_{ijk})_{i,j,k\in I}$ and $(\pi_2^*\theta_{ijk})_{i,j,k\in I}$ coincide as well, since the functions θ_{ijk} are locally constant. Hence, the categories of twisted coherent sheaves $Coh(\mathcal{M},\pi_1^*\theta)$ and $Coh(\mathcal{M},\pi_2^*\theta)$ coincide, and so do $D^b(\mathcal{M},\pi_1^*\theta)$ and $D^b(\mathcal{M},\pi_2^*\theta)$. The morphism π_2 is affine. Consequently, the functor

$$\pi_{2,*} : Coh(\mathcal{M},\pi_1^*\theta) \to Coh(M,\theta)$$

is well defined as is its derived analogue $R\pi_{2,*} : D^b(\mathcal{M},\pi_1^*\theta) \to D^b(M,\theta)$. Any object E in $D^b(\mathcal{M})$ is thus a Fourier-Mukai kernel for an endofunctor $R\pi_{2,*}(L\pi_1^*(\bullet)\otimes E)$ of $D^b(M,\theta)$. The universal Atiyah class (7.3.3) is thus also a natural transformation of endofunctors of $D^b(M,\theta)$.

Definition 7.3.5. Let $\mathcal{B}$ be a θ-twisted sheaf on M. The *Atiyah class*[15]

$$at_{\mathcal{B}} \in \mathrm{Ext}^1(\mathcal{B},\mathcal{B}\otimes\Omega_M) := \mathrm{Hom}(\mathcal{B},\mathcal{B}\otimes\Omega_M[1])$$

of $\mathcal{B}$ is the evaluation of the natural transformation (7.3.3) on the object $\mathcal{B}$.[16]

The *first jet sheaf* of $\mathcal{B}$ is the θ-twisted sheaf $j^1(\mathcal{B}) := \pi_{2,*}(\pi_1^*\mathcal{B})$. Note that if $\mathcal{B}$ is locally free, then $at_{\mathcal{B}}$ is the extension class of the short exact sequence of θ-twisted sheaves

$$0 \to \mathcal{B}\otimes\Omega_M \to j^1(\mathcal{B}) \to \mathcal{B} \to 0$$

associated to the short exact sequence (7.3.2) of Fourier-Mukai kernels in $D^b(\mathcal{M})$. The Atiyah class of a complex of locally free sheaves is defined in [HL, Sec 10.1] and the definition goes through for complexes $E^\bullet$ of θ-twisted locally free sheaves, hence for objects in $D^b(M,\theta)$. One has the identity

$$at_{E\otimes F} = at_E \otimes id_F + id_E \otimes at_F, \tag{7.3.4}$$

for objects E, F in $D^b(M,\theta)$ (see [HL, Sec 10.1]).

Definition 7.3.6. Let $\mathcal{B}$ be a μ_ρ-twisted sheaf. Define $c_1(\mathcal{B}) \in H^1(M,\mathcal{O}_M)$ as $tr(at_{\mathcal{B}})$ and let $ch(\mathcal{B}) \in \oplus_q H^q(M,\Omega^q_M)$ be the trace of the exponential Atiyah class $\exp(at_{\mathcal{B}})$. If $r := \mathrm{rank}(\mathcal{B})$ is non-zero, set $\kappa(\mathcal{B}) := ch(\mathcal{B})\exp(-c_1(\mathcal{B})/r)$.

Note that $c_1(\mathcal{B})$ belongs to the image in $H^1(M,\Omega^1_M)$ of the Neron-Severi group tensored with $\mathbb{Q}$. It suffices to show it for $\mathcal{B}$ of non zero rank r. The object $\mathcal{B}^{\otimes\rho}$ is untwisted, and $c_1(\mathcal{B}^{\otimes\rho}) = \rho r^{\rho-1}c_1(\mathcal{B})$, by Equation (7.3.4). Similarly, $ch(\mathcal{B})$ is a rational class, since $ch(\mathcal{B}^{\otimes\rho}) = ch(\mathcal{B})^\rho$ and so $ch(\mathcal{B})$ is the ρ-th root with constant term r of the Chern character $ch(\mathcal{B}^{\otimes\rho})$.

[15]The Atiyah class for more general twisted sheaves is defined in [Li, Sec. 6.5.1].

[16]This agrees with our sign convention in footnote 14, see [At, Theorem 5].

Lemma 7.3.7. *Let $\mathcal{B}$ be a coherent sheaf of positive rank r twisted by a cocycle θ with coefficients in μ_r and with a trivial determinant line bundle. Assume that the image of θ in $H^2(M, \mathcal{O}_M^\times)$ is trivial, so that $\mathcal{B} \cong \mathcal{E} \otimes \mathcal{L}$, where $\mathcal{E}$ is an untwisted rank r coherent sheaf and $\mathcal{L}$ is a rank 1 locally free θ twisted sheaf. Then the following equality holds*

$$at_{\mathcal{B}} = at_{\mathcal{E}} - \frac{c_1(\mathcal{E})}{r} \otimes id_{\mathcal{E}}, \tag{7.3.5}$$

where $\frac{c_1(\mathcal{E})}{r}$ is regarded as a class in $H^1(M, \Omega^1_M)$.

Proof. Equation (7.3.4) yields the equality $at_{\mathcal{B}} = at_{\mathcal{E}} \otimes id_{\mathcal{L}} + id_{\mathcal{E}} \otimes at_{\mathcal{L}}$ and $at_{\mathcal{L}} = c_1(\mathcal{L})$, which is $-\frac{c_1(\mathcal{E})}{r}$, since $\det(\mathcal{B})$ is trivial. □

Let $\mathcal{B}$ be a μ_r-twisted sheaf of positive rank r of trivial determinant. In that case $\kappa(\mathcal{B}) = ch(\mathcal{B})$. We will see in Remark 9.3.10 that Diagram (7.2.2) remains commutative when at_B is replaced with $at_{\mathcal{B}}$ to define $\sigma_{\mathcal{B}}$, and the class $\kappa(B)$ is replaced with the characteristic class $\kappa(\mathcal{B})$.

Definition 7.3.8. A projective bundle B (or a twisted sheaf $\mathcal{B}$, twisted by a Čech 2-cocycle with coefficients in μ_r) is said to be *semiregular*, if the semiregularity map σ_B (resp. $\sigma_{\mathcal{B}}$) is injective.

The following is the analogue of [BF1, Th. 5.1] for projective bundles and for twisted sheaves. It should follow from [Pr, Remark 2.26], though one needs to reconcile the different terminology. In [Pr] the semiregularity map takes values in Deligne cohomology. See also [HP, Theorem 14.4] for an application of Pridham's semiregularity theorem for perfect twisted objects.

Conjecture 7.3.9. *Let $\pi : \mathcal{M} \to S$ be a deformation of a smooth complex projective variety M_0 over a smooth germ $(S, 0)$ and set $M_s := \pi^{-1}(s)$ for $s \in S$. Assume that $\mathcal{B}$ is a semiregular rank r coherent sheaf over M_0, twisted by a cocycle with coefficients in μ_r, such that for all p the class $ch_p(\mathcal{B})$ extends to a horizontal section of $R^{2p}\pi_*\mathbb{Q}$, which belongs to the direct summand $R^p\pi_*\Omega^p_\pi$ under the Hodge decomposition. Then $\mathcal{B}$ extends to a twisted coherent sheaf over $\pi^{-1}(U)$ for some open analytic neighborhood U of 0 in S.*

Note that $ch(\mathcal{B})$ remains of Hodge type, if and only if both $c_1(\mathcal{B})$ and $\kappa(\mathcal{B})$ remain of Hodge type. The locus where $\kappa(\mathcal{B})$ remains of Hodge type contains the one where $ch(\mathcal{B})$ does. If $\kappa(\mathcal{B})$ remains of Hodge type over S, but $ch(\mathcal{B})$ does not, we can use Construction 7.3.3 to replace $\mathcal{B}$ with a twisted sheaf $\mathcal{B}'$ with trivial determinant, which is the tensor product of $\mathcal{B}$ with a twisted line bundle. The sheaf $\mathcal{B}'$ satisfies $\kappa(\mathcal{B}) = \kappa(\mathcal{B}')$ and $\mathcal{B}'$ satisfies the hypotheses of Conjecture 7.3.9, since $\kappa(\mathcal{B}') = ch(\mathcal{B}')$.

7.4. **The Semiregularity Theorem for μ_r-twisted sheaves on abelian varieties.** In this section we prove Conjecture 7.3.9 in case π is a family of abelian varieties. The proof establishes Conjecture 7.3.9, more generally, whenever the Brauer class $[\theta] \in H^2(M_0, \mu_r)$ of $\mathcal{B}$ is the restriction of the Brauer class of a projective bundle over $\pi^{-1}(U)$, for some open neighborhood U of 0 in S. Note[17] that if $\mathcal{M}$ is quasi projective and $[\theta]$

[17] I thank Nick Addington for this comment.

is the restriction of a Brauer class on $\mathcal{M}\times_S \tilde{S}$, for some finite étale cover $\tilde{S}\to S$, then such a projective bundle exists, by Gabber's theorem [dJ].

7.4.1. *Construction of a projective bundle.* Let X be an abelian variety. The composition

$$H^2(X,\mu_r)\stackrel{\cong}{\to} H^2(X,\frac{1}{r}\mathbb{Z}/\mathbb{Z})\to H^2(X,\mathbb{Q}/\mathbb{Z}) \tag{7.4.1}$$

is injective, where the left arrow is the isomorphism induced by the inverse of the sheaf isomorphism $\exp(2\pi i(\bullet)):\frac{1}{r}\mathbb{Z}/\mathbb{Z}\to\mu_r$ and the right arrow is induced by the inclusion $\frac{1}{r}\mathbb{Z}\subset\mathbb{Q}$. The injectivity of the right arrow follows from the snake lemma applied to

$$\begin{tikzcd} 0 \arrow[r] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "="'] & H^2(X,\frac{1}{r}\mathbb{Z}) \arrow[r] \arrow[d] & H^2(X,\frac{1}{r}\mathbb{Z}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathbb{Q}) \arrow[r] & H^2(X,\mathbb{Q}/\mathbb{Z}) \arrow[r] & 0, \end{tikzcd}$$

where the right exactness of the horizontal rows follows from the Universal Coefficient Theorem and the torsion freeness of $H^*(X,\mathbb{Z})$, which implies also the injectivity of the middle vertical arrow. Given a $\mathbb{P}^{r-1}$ bundle B over X we denote by $[B]\in H^1(X,PGL(r,\mathcal{O}_X))$ its isomorphism class and by $\bar{c}_1(B)\in H^2(X,\mu_r)$ the image of $[B]$ via the connecting homomorphism of the short exact sequence (7.3.1). Denote by

$$\delta(B)$$

the image of $\bar{c}_1(B)$ in $H^2(X,\mathbb{Q}/\mathbb{Z})$ via (7.4.1).

Lemma 7.4.1. *Let $\pi:\mathcal{X}\to S$ be a smooth and proper morphism of smooth analytic spaces all of which fibers are connected abelian varieties. Let X be the fiber of π over $0\in S$. Given a class $\delta_0\in H^2(X,\mathbb{Q}/\mathbb{Z})$, there exists an open neighborhood U of 0 in S and a holomorphic projective bundle $p:\mathbb{P}\to\pi^{-1}(U)$, such that $\delta(\mathbb{P}_{|X})=\delta_0$.*

Proof. Step 1: Assume first that δ_0 is the image of a class $\frac{1}{r}\tilde{\delta}_0$ via the natural homomorphism $H^2(X,\mathbb{Q})\to H^2(X,\mathbb{Q}/\mathbb{Z})$, where $\tilde{\delta}_0\in H^2(X,\mathbb{Z})$ is a primitive class of Hodge type $(1,1)$. Then there exists a simple semi-homogeneous vector bundle Q over X with $\delta(\mathbb{P}(Q))=\delta_0$, by [Mu4, Theorem 7.11(1)]. The rank $r(Q)$ of Q is divisible by r and divides r^n, where n is the dimension of X. Let $\Sigma(Q)\subset\hat{X}$ be the subgroup $\{L\ :\ Q\otimes L\cong Q\}$. Choose an origin for X and denote by X_r the subgroup of X of r-torsion points. We have the short exact sequence

$$0\to X_r\cap K(\tilde{\delta}_0)\to X_r\to\Sigma(Q)\to 0,$$

where $K(\tilde{\delta}_0)$ is the kernel of the homomorphism $\phi_D:X\to\hat{X}$ associated to a line-bundle D with $c_1(D)=\tilde{\delta}_0$, by [Mu4, Theorem 7.11(4)]. Furthermore, $\mathcal{E}nd(Q)\cong\oplus_{L\in\Sigma(Q)}L$, by [Mu4, Proposition 7.1].

The subgroup $\Sigma(Q)$ consists of isomorphism classes of line bundles on X. However, we may and do regard $\Sigma(Q)$ as subgroup of autoequivalences of $D^b(X)$ endowed with a linearization, i.e., a choice of a line-bundle representing each isomorphism class in $\Sigma(Q)$ together with isomorphisms between the composite functor $L_1\otimes(L_2\otimes(\bullet))$ and $L_3\otimes(\bullet)$, for all $L_1, L_2, L_3 \in \Sigma(Q)$ such that L_3 is isomorphic to $L_1 \otimes L_2$, and these isomorphisms of functors satisfy the natural associativity axioms. Such a linearization is provided by conjugating the action on $D^b(\hat{X})$ of the group $\Sigma(Q)$ of translation automorphisms of $\hat{X}$ via the equivalence $\Phi_{\mathcal{P}} : D^b(X) \rightarrow D^b(\hat{X})$ with the Poincare line bundle $\mathcal{P}$ as a Fourier-Mukai kernel. Alternatively, it is provided by the Appell-Humbert theorem [BL, Theorem 2.2.3].

Set $\mathcal{A} := \oplus_{L\in\Sigma(Q)} L$. Let $a : \mathcal{A}\otimes\mathcal{A} \rightarrow \mathcal{A}$ be a homomorphism of $\mathcal{O}_X$-modules. Denote by

$$\lambda, \rho : \mathcal{A} \rightarrow \mathcal{E}nd(\mathcal{A}, \mathcal{A})$$

the homomorphism $\lambda(s) = a(s, (\bullet))$ and $\rho(s) = a((\bullet) \otimes s)$. Then a is the multiplication for the structure of an Azumaya algebra on $\mathcal{A}$, if and only if a endows $\mathcal{A}$ with the structure of a sheaf of associative algebras with a unit, and $\lambda \otimes \rho : \mathcal{A} \otimes \mathcal{A}^\circ \rightarrow \mathcal{E}nd(\mathcal{A})$ is an isomorphism of sheaves of algebras (see [Mi1, Proposition IV.2.1]). Here $\mathcal{A}^\circ$ is $\mathcal{A}$ endowed with the multiplication a° obtained by composing a with the transposition of the tensor factors of $\mathcal{A} \otimes \mathcal{A}$. The condition that $\lambda \otimes \rho$ is an isomorphism of sheaves of algebras means that it is an isomorphism of $\mathcal{O}_X$-modules satisfying

$$(\lambda \otimes \rho)(s_2 \otimes t_2) \circ (\lambda \otimes \rho)(s_1 \otimes t_1) = (\lambda \otimes \rho)(a(s_2 \otimes s_1) \otimes a(t_1 \otimes t_2))$$

for local sections s_1, s_2 of $\mathcal{A}$ and t_1, t_2 of $\mathcal{A}^\circ$.

Given $L_1, L_2 \in \Sigma(Q)$, let the sheaf homomorphism $e_{L_2,L_1} : L_2 \otimes L_1^{-1} \rightarrow \mathcal{E}nd(\mathcal{A}, \mathcal{A})$ send a section s of $L_2 \otimes L_1^{-1}$ to the homomorphism mapping the direct summand L_1 of $\mathcal{A}$ to the direct summand L_2 by tensorization by s and annihilating all other direct summands of $\mathcal{A}$. There exist complex numbers a_{L_1,L_2}, such that

$$\lambda = \sum_{L_1,L_2\in\Sigma(Q)} a_{L_1,L_2} e_{L_2,L_1}.$$

Here we use the linearization of the action of $\Sigma(Q)$ of $D^b(A)$, which provides an isomorphism between $L_2 \otimes L_1^{-1}$ and a direct summand of $\mathcal{A}$. Consider the composition

$$L_3 \otimes L_1^{-1} \stackrel{\cong}{\rightarrow} (L_3 \otimes L_2^{-1}) \otimes (L_2 \otimes L_1^{-1}) \stackrel{e_{L_3,L_2}\otimes e_{L_2,L_1}}{\longrightarrow} \mathcal{E}nd(\mathcal{A}) \otimes \mathcal{E}nd(A) \rightarrow \mathcal{E}nd(\mathcal{A}),$$

where the right arrow is multiplication in $\mathcal{E}nd(\mathcal{A})$. The above composition is equal to e_{L_3,L_1} and is independent of a. Thus, the conditions that λ and ρ are homomorphisms of sheaves of algebras and that $\lambda \otimes \rho$ is an isomorphism of sheaves of algebras are all expressed in terms of universal equations among the constants a_{L_1,L_2}. These equations depend on the group $\Sigma(Q)$ as an abstract abelian group, but are independent of the complex structure of X and of the embedding of $\Sigma(Q)$ as a subgroup of $\hat{X}$.

The subgroup $\Sigma(Q)$ of $\hat{X}$ deforms with X canonically as a fiber of a local system Σ, which is trivial over a simply connected open neighborhood U of 0 in S. Hence, The locally free sheaf $\mathcal{A}$ deforms with X to a locally free sheaf over $\pi^{-1}(U)$. Indeed, it is simply the pushforward to the first factor of the restriction to $\mathcal{X} \times_U \Sigma$ of a relative Poincaré line bundle over $\mathcal{X} \times_U \mathrm{Pic}^0(\mathcal{X}/U)$. The constants a_{L_i,L_j} vary in the local system $\mathbb{C}[\Sigma^2]$ over U of functions from the fibers of $\Sigma \times_U \Sigma$ to $\mathbb{C}$. We get a locally trivial fibration $Azu \subset \mathbb{C}[\Sigma^2]$ by the space of solutions to the equations defining the structure of an Azumaya algebra. The fibration Azu is trivial over every subset where the local system Σ is trivial, in particular over the simply connected neighborhood U. Choosing a flat section of $\mathbb{C}[\Sigma^2]$ with value[18] a over $0 \in U$ equal to the multiplication in $\mathcal{E}nd(Q)$, we get a section of Azu and hence also the desired deformation of $(\mathcal{A}, a)$ via an Azumaya algebra over $\pi^{-1}(U)$. We get the holomorphic projective bundle $\mathbb{P}$ over $\pi^{-1}(U)$, by the bijection between isomorphism classes of Azumaya algebras and projective bundles [Mi1, Proposition IV.2.3].

Step 2: If the class δ_0 does not admit a lift to $\frac{1}{r}\tilde{\delta}_0 \in H^{1,1}(X, \mathbb{Z})$, then we can choose an auxiliary family of abelian varieties $\mathcal{X}' \to S'$ over a simply connected analytic base S' with one fiber X and another X', such that the parallel transport δ_1 of δ_0 to the fiber X' does admit such a lift. The construction in Step 1 gives rise to a projective bundle over $\mathcal{X}'$ whose restriction $\mathbb{P}_0$ to X has class δ_0. Hence, we get an Azumaya algebra $\mathcal{A}$ over X, which is again isomorphic as a locally free sheaf to the direct sum of line bundles in a subgroup Σ_0 of $\hat{X}$. Repeating the construction in Step 1 with Σ_0 instead of $\Sigma(Q)$ we get the desired projective bundle $\mathbb{P}$ over $\pi^{-1}(U)$, for any simply connected open neighborhood U of 0 in S. □

Let $p : \mathbb{P} \to \pi^{-1}(U)$ be as in Lemma 7.4.1. Let $\mathcal{D} \subset \mathbb{P} \times_{\pi^{-1}(U)} \mathbb{P}^*$ be the incidence divisor. Given a point $x \in \mathcal{X}$, the line bundle $\mathcal{O}_{\mathbb{P}\times_{\pi^{-1}(U)}\mathbb{P}^*}(\mathcal{D})$ restricts to the fiber $\mathbb{P}_x \times \mathbb{P}_x^*$ as $\mathcal{O}_{\mathbb{P}_x\times\mathbb{P}_x^*}(1,1)$. Let $\mathcal{Q}_\mathbb{P}$ be the direct image of $\mathcal{O}_{\mathbb{P}\times_{\pi^{-1}(U)}\mathbb{P}^*}(\mathcal{D})$ via the projection from $\mathbb{P}\times_{\pi^{-1}(U)}\mathbb{P}^*$ to the first factor $\mathbb{P}$. Then $\mathcal{Q}_\mathbb{P}$ is a locally free sheaf over $\mathbb{P}$, which restriction to the fiber $\mathbb{P}_x$ is a direct sum of copies of $\mathcal{O}_{\mathbb{P}_x}(1)$ and such that $p_*\mathcal{Q}_\mathbb{P}$ is isomorphic to the Azumaya algebra of $\mathbb{P}$.

7.4.2. *Proof of Conjecture 7.3.9 when $\pi : \mathcal{M} \to S$ is a family of connected abelian varieties.*

Proof. Step 1: Let $\mathcal{B}$ be a rank r semiregular coherent sheaf over M_0 twisted by a Čech 2-cocycle θ with coefficients in μ_ρ, for some ρ dividing r, and with a trivial determinant line bundle. The latter assumption will be dropped in Step 6. Note that we may and do assume that the order of the class $[\theta]$ in $H^2(M_0, \mu_\rho)$ is ρ, by Remark 7.3.2. There exists an open neighborhood U of 0 in S and a projective bundle $p : \mathbb{P} \to \pi^{-1}(U)$, which restricts to M_0 as a projective bundle $\mathbb{P}_0$ with $\bar{c}_1(\mathbb{P}_0) = [\theta] \in H^2(M_0, \mu_\rho)$, by Lemma 7.4.1. Let Q_0 be a lift of $\mathbb{P}_0$ to a θ-twisted locally free sheaf over M_0, such that $\mathbb{P}_0$ is isomorphic to $\mathbb{P}(Q_0)$ (here we use Remark 7.3.2 to keep the coefficients in μ_ρ, rather than in $\mu_{\mathrm{rank}(Q_0)}$). We may and do assume that $\det(Q_0)$ is trivial, by Example 7.3.1. The pullback $p_0^*Q_0^*$ admits a tautological quotient rank one $p_0^*\theta^{-1}$-twisted locally free sheaf $\mathcal{O}_{\mathbb{P}(Q_0)}(1)$. Note that $(\mathcal{O}_{\mathbb{P}(Q_0)}(1))^{\otimes\rho}$ is untwisted, as θ has coefficients in μ_ρ. We denote the line bundle $(\mathcal{O}_{\mathbb{P}(Q_0)}(1))^{\otimes\rho k}$ by $\mathcal{O}_{\mathbb{P}(Q_0)}(\rho k)$, for any integer k. Let $\tilde{E}_0$ be[19] the

[18] Note that the projective bundle $\mathbb{P}(Q)$ is infinitesimally rigid over X and its automorphism group is trivial, by [Mu4, Proposition 5.9 and Lemma 6.7]. It follows that the connected component of a in the fiber of Azu over 0 is a single point, by the bijection between isomorphism classes of projective bundles and of Azumaya algebras [Mi1, Proposition IV.2.3].

[19] In the special case where $\mathcal{B}$ is obtained from an untwisted reflexive sheaf E, as in Construction 7.3.3, we can define $\tilde{E}_0$ more directly as follows. Let Q be a simple semihomogeneous bundle with $c_1(Q)/\mathrm{rank}(Q) = c_1(E)/\mathrm{rank}(E)$. We can further choose Q, so that $\det(E \otimes Q^*)$ is trivial. Let $\mathbb{P}$ be the extension of $\mathbb{P}(Q)$ over $\pi^{-1}(U)$ provided by Lemma 7.4.1 and set $\tilde{E}_0 := (p_0^*E) \otimes \mathcal{O}_{\mathbb{P}(Q)}(1)$.

corresponding (untwisted) tautological quotient sheaf $(p_0^*\mathcal{B})\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(1)$ of $p_0^*(\mathcal{B}\otimes Q_0^*)$. Then $p_{0,*}\tilde{E}_0$ is isomorphic to the untwisted sheaf $\mathcal{B}\otimes Q_0^*$ with trivial determinant.

Step 2: We prove next that $c_1(\tilde{E}_0)\in H^2(\mathbb{P}_0,\mathbb{Z})$ remains of Hodge type $(1,1)$ over each fiber of $\pi\circ p:\mathbb{P}\rightarrow U$. The line bundle $\det(\tilde{E}_0)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(Q_0)}(r)$, since $\det(\mathcal{B})$ is trivial. The locally free sheaf $p_0^*Q_0\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(1)$ is untwisted. The line bundle $\det((p_0^*Q_0)\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(1))$ is isomorphic to $\mathcal{O}_{\mathbb{P}(Q_0)}(\mathrm{rank}(Q_0))$, since $\det(Q_0)$ is trivial. The line bundle $\mathcal{O}_{\mathbb{P}(Q_0)}(\mathrm{rank}(Q_0))\cong\det((p_0^*Q_0)\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(1))$ is isomorphic to the restriction of $\det(\mathcal{Q}_{\mathbb{P}})$ to $\mathbb{P}_0$. Hence, $c_1(\mathcal{O}_{\mathbb{P}(Q_0)}(\rho))$ remains of Hodge type $(1,1)$ over every fiber of $p\circ\pi:\mathbb{P}\rightarrow U$. Consequently, so does $c_1(\tilde{E}_0)$.

Step 3: The Chern character $ch(\tilde{E}_0)$ is equal to $\kappa(\tilde{E}_0)\exp(c_1(\tilde{E}_0)/r)$, which is equal to $p^*\kappa(\mathcal{B})\exp(c_1(\tilde{E}_0)/r)$. It remains of Hodge type over every fiber of $\pi\circ p:\mathbb{P}\rightarrow U$, since $\kappa(\mathcal{B})$ does, by assumption, and $c_1(\tilde{E}_0)$ does, by Step 2.

Step 4: We prove next that the sheaf $\tilde{E}_0$ is semiregular. The homomorphism p_0^* : $\mathrm{Ext}^2(\mathcal{B},\mathcal{B})\rightarrow\mathrm{Ext}^2(p_0^*\mathcal{B},p_0^*\mathcal{B})$ is an isomorphism, since $Rp_{0,*}\mathcal{O}_{\mathbb{P}_0}$ is isomorphic to $\mathcal{O}_{M_0}$ and so

$$\mathrm{Ext}^2(p_0^*\mathcal{B},p_0^*\mathcal{B})\cong H^2(M_0,R\mathcal{H}om(\mathcal{B},\mathcal{B})\otimes Rp_{0,*}\mathcal{O}_{\mathbb{P}_0})\cong H^2(M_0,R\mathcal{H}om(\mathcal{B},\mathcal{B}))\cong\mathrm{Ext}^2(\mathcal{B},\mathcal{B}).$$

Set $\lambda:=c_1(\tilde{E}_0)$. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^2(\mathcal{B},\mathcal{B}) \arrow[r, "p_0^*"] \arrow[d, "\sigma_{\mathcal{B}}"] & \mathrm{Ext}^2(p_0^*\mathcal{B},p_0^*\mathcal{B}) \arrow[r, "\cong"] \arrow[d, "\sigma_{p_0^*\mathcal{B}}"] & \mathrm{Ext}^2(\tilde{E}_0,\tilde{E}_0) \arrow[d, "\sigma_{\tilde{E}_0}"'] \\
\oplus_{q=0}^{2n-2}H^{q+2}(\Omega^q_{M_0}) \arrow[r, "p_0^*"'] & \oplus_{q=0}^{2n+2r-4}H^{q+2}(\Omega^q_{\mathbb{P}_0}) \arrow[r, "\cup(\exp(\lambda/r))"'] & \oplus_{q=0}^{2n+2r-4}H^{q+2}(\Omega^q_{\mathbb{P}_0})
\end{tikzcd}$$

The commutativity of the right square follows from Equation (7.3.5). The top right isomorphism is induced by the natural isomorphism $R\mathcal{H}om(p_0^*\mathcal{B},p_0^*\mathcal{B})\cong R\mathcal{H}om(\tilde{E}_0,\tilde{E}_0)$. The top horizontal homomorphisms are isomorphisms. The bottom horizontal homomorphisms are injective, and the left vertical homomorphism is the semiregularity map $\sigma_{\mathcal{B}}$, which is assumed to be injective. Hence, the semiregularity map $\sigma_{\tilde{E}_0}$ is injective as well.

Step 5: The Semiregularity Theorem [BF1, Th. 5.1] implies that the sheaf $\tilde{E}_0$ extends to a coherent sheaf $\tilde{E}$ over $\mathbb{P}$, possibly after replacing the open neighborhood U of 0 in S by a smaller open neighborhood. We may further shrink U, so that the fibration $\pi^{-1}(U)\rightarrow U$ is topologically trivial and thus extend the cocycle θ in $Z^2(\mathcal{U},\mu_\rho)$ to a Čech 2-cocycle θ' over $\pi^{-1}(U)$, which restricts to M_0 as θ. Then there exists a θ'-twisted sheaf $\mathcal{Q}'$ over $\pi^{-1}(U)$, such that the restriction of $\mathcal{Q}'$ to M_0 is isomorphic to Q_0 and $\mathbb{P}(\mathcal{Q}')$ is isomorphic to $\mathbb{P}$. We get the $p^*\theta'$-twisted tautological line subbundle $\mathcal{O}_{\mathbb{P}(\mathcal{Q}')}(-1)$ of $p^*\mathcal{Q}'$, which restricts to $\mathbb{P}_0$ as $\mathcal{O}_{\mathbb{P}(Q_0)}(-1)$. We have the isomorphism $p_{0,*}\left(\tilde{E}_0\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(-1)\right)\cong\mathcal{B}$ and the vanishing $Rp^i_{0,*}\left(\tilde{E}_0\otimes\mathcal{O}_{\mathbb{P}(Q_0)}(-1)\right)=0$, for $i>0$. Hence, we may assume that $Rp^i_*\left(\tilde{E}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{Q}')}(-1)\right)$ vanishes, for $i>0$, by upper-semi-continuity, possibly after shrinking the open neighborhood U further. Consequently,

$p_*\left(\tilde{E}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{Q}')}(-1)\right)$ is a coherent θ'-twisted sheaf over $\pi^{-1}(U)$, flat over U, extending $\mathcal{B}$.

Step 6: Finally, we drop the assumption that $\det(\mathcal{B})$ is trivial. Let $\mathcal{F}$ be a coherent sheaf with trivial determinant, twisted by a cocycle α with coefficients in μ_r, and L a line bundle twisted by a cocycle β with coefficients in μ_r, so that $\theta=\alpha\beta$ and $\mathcal{B}$ is isomorphic to $\mathcal{F}\otimes L$. Such $\mathcal{F}$ and L are constructed in Construction 7.3.3. There exists an open subset U of S, a cocycle α' with coefficients in μ_r, and an α'-twisted coherent sheaf $\mathcal{F}'$ over $\pi^{-1}(U)$ extending $\mathcal{F}$, by Step 5. It thus remains to extend L. The untwisted line bundle L^r is isomorphic to $\det(\mathcal{B})$, and so it extends to a line bundle N over $\pi^{-1}(U)$, by the assumption that $c_1(\mathcal{B})$ remains of Hodge type, possibly after shrinking U. Let L' be a β'-twisted r-th root of N, where β' is a 2-cocycle with coefficients in μ_r, as in Construction 7.3.3. The restriction $\bar{L}'$ of L' to M_0 satisfies $(\bar{L}')^r\cong L^r$. Hence, the transition functions of $(\bar{L}')^{-1}\otimes L$ in the open covering in Construction 7.3.3 form a 1-cochain γ with coefficients in μ_r. Let γ' be an extension of γ to a 1-cochain with coefficients in μ_r over $\pi^{-1}(U)$ using the topological triviality of π, as in the previous step. Multiplying the transition functions of L' by γ' we obtain a twisted line bundle extending L. $\square$

8. Secant sheaves on abelian threefolds

In Section 8.1 we consider examples of ideal sheaves twisted by a line-bundle, which are secant sheaves over abelian threefolds and fourfolds. In Section 8.2 we consider the ideal sheaf $F:=\mathcal{I}_{\cup_{i=1}^{d+1}C_i}$ of $d+1$ translates C_i of the Abel-Jacobi image of a non-hyperelliptic curve C of genus 3 in its Jacobian $X=\mathrm{Pic}^2(C)$. We show that the obstruction map $ob_F: HT^2(X)\rightarrow \mathrm{Ext}^2(F,F)$ has rank 6, and so its kernel is a 9-dimensional subspace of unobstructed first order deformations of the pair (X,F).

In Section 8.3 we set $F_1:=\mathcal{I}_{\cup_{i=1}^{d+1}C_i}(\Theta)$ and $F_2:=\mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}(\Theta)$, where Σ_i are images under the natural involution $L\mapsto \omega_C\otimes L^{-1}$ of $\mathrm{Pic}^2(C)$ of translates of the Abel-Jacobi image of C. Let the object $E:=\Phi(F_1\boxtimes F_2)$ be the image of the outer tensor product of F_1 and F_2 via Orlov's equivalence $\Phi: D^b(X\times X)\rightarrow D^b(X\times\hat{X})$. We show that the kernel of $ob_E: HT^2(X\times\hat{X})\rightarrow \mathrm{Ext}^2(E,E)$ is equal to the kernel of the homomorphism $ch(E): HT^2(E)\rightarrow H^*(X\times\hat{X},\mathbb{C})$ via the $HT^*(X\times\hat{X})$-module structure of $H^*(X\times\hat{X},\mathbb{C})$ and the action of $HT^*(X\times\hat{X})$ on $ch(F)\in HH^*(X\times\hat{X},\mathbb{C})$.

In Section 8.4 we show that the isomorphism $\Phi^{HT}: HT^2(X\times X)\rightarrow HT^2(X\times\hat{X})$ maps the "diagonally" embedded $\ker(ob_{F_1})\subset HT^2(X)$ to a 9-dimensional subspace of first order commutative and gerby deformations of $X\times\hat{X}$.

8.1. **Examples of secant sheaves.** Let C be a non-hyperelliptic curve of genus 3. Set $X:=\mathrm{Pic}^2(C)$ and let $\Theta\subset X$ be the canonical divisor. The natural morphism $C^{(2)}\rightarrow\Theta$ is an isomorphism. Indeed, the morphism is injective, Θ is smooth, by Riemann's Singularity Theorem [BL, Theorem 11.2.5], and so the morphism is an isomorphism by Zariski's Main Theorem. Let $AJ: C\rightarrow \mathrm{Pic}^1(X)$ be the Abel-Jacobi morphism. Given a point $t\in\mathrm{Pic}^1(X)$ denote by $C_t\subset X$ the translate of $AJ(C)$ by t. Denote by $[pt]\in H^6(X,\mathbb{Z})$ the class Poincaré-dual to a point. Given a subvariety Z of X, denote by $[Z]$ the class in $H^*(X,\mathbb{Z})$ Poncaré-dual to Z. Then $[\Theta]^3/6=[pt]$ and

$$[C_t]=[\Theta]^2/2,$$

by Poncaré's formula [BL, Sec. 11.2]. We denote $[\Theta]$ by Θ as well.

Let d be a positive integer. Set $\alpha:=1-\frac{d}{2}\Theta^2$ and $\beta:=\Theta-d[pt]$. We have

$$\exp(\sqrt{-d}\Theta)=1+\sqrt{-d}\Theta-\frac{d}{2}\Theta^2-d\sqrt{-d}[pt]=\left(1-\frac{d}{2}\Theta^2\right)+\sqrt{-d}\left(\Theta-d[pt]\right)=\alpha+\sqrt{-d}\beta.$$

Cup product with $\exp(\sqrt{-d}\Theta)$ is an automorphism of $S_K:=H^*(X,K)$, which belongs to the image of $m:\mathrm{Spin}(V_K)\rightarrow GL(S_K)$, where m is given in (2.1.3). Now $1\in H^0(X,\mathbb{Z})$ is an even pure spinor, hence, so is $\exp(\sqrt{-d}\Theta)$. Let t_i, $1\leq i\leq d+1$, be distinct points of $\mathrm{Pic}^1(C)$ and set $C_i:=C_{t_i}$.

Lemma 8.1.1. *Assume that C_i, $1\leq i\leq d+1$, are pairwise disjoint. Then the following equality holds*

$$ch\left(\mathcal{I}_{\cup_{i=1}^{d+1}C_i}\otimes\mathcal{O}_X(\Theta)\right)=1+\Theta-\frac{d}{2}\Theta^2-d[pt]=\alpha+\beta.$$

Consequently, $ch\left(\mathcal{I}_{\cup_{i=1}^{d+1}C_i}\otimes\mathcal{O}_X(\Theta)\right)$ and $ch\left(R\mathcal{H}om(\mathcal{I}_{\cup_{i=1}^{d+1}C_i}\otimes\mathcal{O}_X(\Theta),\mathcal{O}_X)\right)$ both belong to the secant line to the spinor variety through the pure spinor $\exp(\sqrt{-d}\Theta)$ and its complex conjugate $\exp(-\sqrt{-d}\Theta)$.

Proof. The equality $ch(\mathcal{O}_{C_i})=\Theta^2/2-2[pt]$ holds, since $\chi(\mathcal{O}_{C_i})=-2$. Hence, given n disjoint translates C_i of $AJ(C)$, we get

$$ch(\mathcal{I}_{\cup_{i=1}^nC_i})=1-n\left(\Theta^2/2-2[pt]\right)=1-\frac{n}{2}\Theta^2+2n[pt].$$

Then

$$\begin{aligned} ch(\mathcal{I}_{\cup_{i=1}^nC_i}\otimes\mathcal{O}_X(k\Theta)) &= (1-\frac{n}{2}\Theta^2+2n[pt])(1+k\Theta+\frac{k^2}{2}\Theta^2+k^3[pt]) \\ &= 1+k\Theta+\frac{(k^2-n)}{2}\Theta^2+(k^3-3kn+2n)[pt]). \end{aligned}$$

Taking $k=1$ and $n=d+1$ we get the desired equality. $\square$

Example 8.1.2. Let X be an abelian surface, let F be a coherent sheaf with $w:=ch(F)$ satisfying$(w,w)_S<0$, and let $h\in H^{ev}(X,\mathbb{Z})$ be an algebraic class, such that $(h,w)_S=0$ and $(h,h)_S<0$. Then $\mathrm{span}_{\mathbb{Q}}\{w,h\}$ is a secant to the spinor variety inducing complex multiplication by the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, where $d=\frac{(w,w)(h,h)}{4}$, by [M2, Prop. 1.7]. In particular, F is a $\mathbb{Q}(\sqrt{-d})$-secant sheaf.

Example 8.1.3. When X is an abelian n-fold, $n\geq 4$, Lemma 8.1.1 can be generalized to produce secant sheaves with a secant line inducing complex multiplication by $\mathbb{Q}(\sqrt{-d})$ as follows. Let C be a Brill-Noether generic curve of genus n and set $X:=\mathrm{Pic}^{n-1}(C)$. Denote by W_k the Abel-Jacobi image of $C^{(k)}$ in $\mathrm{Pic}^k(C)$ and by $[W_k]$ the class of any translate of W_k in X. Then $[W_k]=\Theta^{n-k}/(n-k)!$, by Poincaré's formula. Let t_{jk} be generic points in $\mathrm{Pic}^{n-k-1}(C)$. Let the subscheme Z of X be the union

$$Z:=\cup_{k=0}^{n-2}\cup_{j=1}^{a_k}\tau_{t_{jk}}(W_k). \tag{8.1.1}$$

Let $F := \mathcal{I}_Z(a_{n-1}\Theta)$ be the tensor product of the ideal sheaf of Z with $\mathcal{O}_X(a_{n-1}\Theta)$. Let α be the real part of $\exp(\sqrt{-d}\Theta)$ and $\sqrt{d}\beta$ its imaginary part.

$$\begin{array}{rcl} \alpha & := & 1 - d[W_{n-2}] + d^2[W_{n-4}] + \cdots + (-d)^{n/2}[pt] \\ \beta & := & \Theta - d[W_{n-3}] + \cdots \end{array}$$

The integers a_k, $0 \leq k \leq n-1$, in (8.1.1) should be chosen to satisfy the equation

$$ch(F) = \alpha + a_{n-1}\beta. \tag{8.1.2}$$

For example, when $n = 4$ we may assume that only the irreducible components of Z in (8.1.1), which are translates of W_2, intersect and every such pair interests at $\Theta^4/4 = 6$ points. Note that $\chi(\mathcal{O}_{W_1}) = \chi(\mathcal{O}_C) = -3$ and $\chi(\mathcal{O}_{W_2}) = 3$. For the latter equality use that C is non-hyperelliptic to conclude that W_2 is isomorphic to $C^{(2)}$, as well as the equalities $h^{0,1}(C^{(2)}) = 4$ and $h^{0,2}(C^{(2)}) = 6$. We get

$$\begin{array}{rcl} ch(\mathcal{O}_{W_1}) & = & \Theta^3/6 - 3[pt], \\ ch(\mathcal{O}_{W_2}) & = & \Theta^2/2 - \Theta^3/3 + 3[pt] \quad \text{(see Lemma 8.1.4)}, \\ ch(\mathcal{O}_Z) & = & \displaystyle\sum_{k=0}^{2} a_k ch(\mathcal{O}_{W_k}) - 6\binom{a_2}{2}[pt] \\ ch(\mathcal{I}_Z) & = & 1 - \dfrac{a_2}{2}\Theta^2 + \left[\dfrac{2a_2 - a_1}{6}\right]\Theta^3 + \left[6\binom{a_2}{2} - 3a_2 + 3a_1 - a_0\right][pt], \\ ch(\mathcal{I}_Z(a_3\Theta) & = & 1 + a_3\Theta + \left[\dfrac{a_3^2 - a_2}{2}\right]\Theta^2 + \left[\dfrac{a_3^3 - 3a_3a_2 + 2a_2 - a_1}{6}\right]\Theta^3 + \\ & & \left[a_3^4 - 6a_2a_3^2 + 8a_2a_3 - 4a_1a_3 + 6\binom{a_2}{2} - 3a_2 + 3a_1 - a_0\right][pt] \\ \alpha + a_3\beta & = & 1 + a_3\Theta - \dfrac{d}{2}\Theta^2 - \dfrac{da_3}{6}\Theta^3 + d^2[pt] \end{array}$$

Comparing coefficients in Equation 8.1.2 we get $a_2 = d + a_3^2$, $a_1 = 2(d + a_3^2)(1 - a_3)$,

$$a_0 = 6a_3^4 - 6a_3^3 + 8da_3^2 - 6da_3 + 2d^2.$$

We get a secant ideal sheaf tensored with $\mathcal{O}_X(a_3\Theta)$ for every choice of an integer $a_3 \leq 1$. If we choose $a_3 = 1$, then $a_2 = d + 1$, $a_1 = 0$, and $a_0 = 2d(d+1)$. Once again we can choose Z to be invariant with respect to a subgroup of X of order $d + 1$.

Lemma 8.1.4. *We have $ch(\mathcal{O}_{W_2}) = \Theta^2/2 - \Theta^3/3 + 3[pt]$ in the notation of Example 8.1.3.*

Proof. We have $ch(\mathcal{O}_{W_2}) = \Theta^2/2 + \lambda\Theta^3 + 3[pt]$, for some rational number λ satisfying (8.1.3)

$$\chi(\mathcal{O}_{W_2}(-\Theta)) = \int_X \left(\Theta^2/2 + \lambda\Theta^3 + 3[pt]\right)\left(1 - \Theta + \Theta^2/2 - \Theta^3/6 + \Theta^4/24\right) = 9 - 24\lambda,$$

since $\dim H^{3,3}(X, \mathbb{Q})$ is the rank of the Neron-Severi group of X, which is 1 for generic C. Assume that C is the intersection of a smooth quadric Q and a cubic in $\mathbb{P}^3$. So C has two g^1_3's, associated to the two rulings of Q. Let $q_1 + q_2 + q_3$ be a reduced divisor in one g^1_3, denote by $\mathcal{L} := \omega_C(-q_1 - q_2 - q_3)$ the other g^1_3, and let $p \in C \setminus \{q_1, q_2, q_3\}$

be a point, such that $|\mathcal{L}(-p)|=\{a+b\}$ for distinct points $a,b\in C\setminus\{p\}$. We calculate the intersection $W_2\cap\tau_{p-q_1-q_2}(\Theta)$. The intersection has cohomology class $\Theta^3/2$. It consists of the union of three irreducible components of class $\Theta^3/6$ each. As a subset of $X=\mathrm{Pic}^2(C)$, the intersection consists of classes of effective divisors D on C of degree 2, such that $D+q_1+q_2-p$ is effective. One irreducible component is $C_1:=\tau_p(AJ(C))$.

The complement of C_1 in the intersection consist of divisors D, such that $h^0(\mathcal{O}_C(D+q_1+q_2)=2$ (the inequality $h^0(\mathcal{O}_C(D+q_1+q_2)\leq 2$ follows from the assumption that C is not hyperelliptic). Such divisors D satisfy

$$\omega_C(-D')\cong\mathcal{O}_C(D+q_1+q_2),$$

for some effective divisor D'. So $D+D'$ belongs to $|\omega_C(-q_1-q_2)|$. If q_3 is in the support of D, then $D=q_3+q$, where q is any point of C, since $h^0(\mathcal{O}_C(q_1+q_2+q_3))=2$. Hence, a second irreducible component is $C_2:=\tau_{q_3}(AJ(C))$. The third irreducible component C_3 consists of D, such that $D+q\in|\mathcal{L}|$, for some $q\in C$. So $C_3=\tau_{\mathcal{L}}(-AJ(C))$, where $-AJ(C)$ is a curve in $\mathrm{Pic}^{-1}(C)$.

The curves C_1 and C_2 intersect at the point corresponding to the divisor $p+q_3$. The curves C_1 and C_3 intersect at the two points $\{p+a,p+b\}$. Let P be the plane tangent to Q at q_3. Then $P\cap Q$ consists of two lines through q_3 and $P\cap C=q_1+q_2+2q_3+a'+b'$, for some points a',b' of C, which we may assume distinct, possibly by changing the choice of the divisor $q_1+q_2+q_3$. The curves C_2 and C_3 intersect at the two points $\{q_3+a',q_3+b'\}$. We conclude that

$$\chi(\mathcal{O}_{W_2\cap\tau_{p-q_1-q_2}(\Theta)})=\chi(\mathcal{O}_{C_1})+\chi(\mathcal{O}_{C_2})+\chi(\mathcal{O}_{C_3})-5=-14.$$
$$\chi(\mathcal{O}_{W_2}(-\tau_{p-q_1-q_2}(\Theta)))=\chi(\mathcal{O}_{W_2})-\chi(\mathcal{O}_{W_2\cap\tau_{p-q_1-q_2}(\Theta)})=17.$$

Comparing with (8.1.3) we get $9-24\lambda=17$, so $\lambda=-1/3$. □

8.2. **Secant sheaves on abelian threefolds with a rank 6 obstruction map.** Keep the notation of Lemma 8.1.1. Set $F_1:=\mathcal{I}_{\cup_{i=1}^{d+1}C_i}\otimes\mathcal{O}_X(\Theta)$. Let $P:=\mathrm{span}\{\alpha,\beta\}$ be the rational $\mathbb{Q}(\sqrt{-d})$-secant plane. Assumption 2.4.1 is satisfied, since $(\alpha,\beta)_S=\int_X\tau(\alpha)\cup\beta=\int_X\alpha\cup\beta=-4d\neq 0$. Let h be an ample class in the rank 1 subgroup $H^2(X\times\hat{X},\mathbb{Z})^{\mathrm{Spin}(V)_P}$. Such an ample class h exists, by Proposition 2.4.4.

Lemma 8.2.1. *The rank of $\Phi(F_1\boxtimes F_1)$ is non-zero. The $\mathrm{Spin}(V)_P$-invariant classes h^3 and $\kappa_3(\Phi(F_1\boxtimes F_1))$ are linearly independent.*

Proof. We use the notation of Proposition 6.4.1. We have

$$ch(\Phi(F_1\boxtimes F_1))=\phi(ch(F_1)\boxtimes ch(F_1))=\phi'(ch(F_1)\boxtimes\tau(ch(F_1))).$$

Set $\lambda_1:=\exp(\sqrt{-d}\Theta)$ and $\lambda_2=\bar{\lambda}_1$, so that

$$ch(F_1)=\frac{1}{2}\left[(\lambda_1+\lambda_2)+\frac{1}{\sqrt{-d}}(\lambda_1-\lambda_2)\right]=\frac{1}{2\sqrt{-d}}\left[(1+\sqrt{-d})\lambda_1+(-1+\sqrt{-d})\lambda_2\right]$$

Now, τ interchanges λ_1 and λ_2. Hence,

$$\begin{aligned}ch(F_1)\boxtimes\tau(ch(F_1)) &= \frac{-1}{4d}\Big[[(1+\sqrt{-d})\lambda_1+(-1+\sqrt{-d})\lambda_2]\boxtimes[(-1+\sqrt{-d})\lambda_1+(1+\sqrt{-d})\lambda_2]\Big]\\ &= \frac{d+1}{4d}[\lambda_1\boxtimes\lambda_1+\lambda_2\boxtimes\lambda_2]+\frac{d-1}{4d}[\lambda_1\boxtimes\lambda_2+\lambda_2\boxtimes\lambda_1]+\frac{\sqrt{-d}}{2d}[\lambda_2\boxtimes\lambda_1-\lambda_1\boxtimes\lambda_2]\end{aligned}$$

The rank of $\Phi(F_1 \boxtimes F_1)$ is non-zero, by Proposition 6.4.1(2), since the coefficient of $[\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2]$ is non-zero. We prove that the classes $\kappa(\Phi(F_1 \boxtimes F_1))$ and h^3 are linearly independent, by contradiction. Assume otherwise. Then $\kappa(\Phi(F_1 \boxtimes F_1))$ belongs to the subring generated by powers of h and is thus $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho$-invariant, by Lemma 2.2.7. Hence, $ch(\Phi(F_1 \boxtimes F_1))$ is $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho'$-invariant, by Lemma 6.2.6. It follows that $ch(F_1) \boxtimes \tau(ch(F_1))$ is invariant under $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$, by the $\mathrm{Spin}(V)$-ρ'-equivariance of $\phi \circ (id \otimes \tau)$. But the last two summands $\lambda_1 \boxtimes \lambda_2 + \lambda_2 \boxtimes \lambda_1$ and $\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2$ in the displayed formula above are $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant, while the first summand is a scalar multiple of $\lambda_1 \boxtimes \lambda_1 + \lambda_2 \boxtimes \lambda_2$ and is a not $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant, by Lemma 2.2.7 and Proposition 6.4.1(1). Hence, $ch(F_1) \boxtimes \tau(ch(F_1))$ is not $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant, since the coefficient of $\lambda_1 \boxtimes \lambda_1 + \lambda_2 \boxtimes \lambda_2$ is non zero. A contradiction. $\square$

Set $F := \mathcal{I}_{\cup_{i=1}^n C_i}$. Let $U \subset (\mathrm{Pic}^0(C))^n$ be the Zariski open subset of points $(\ell_1, \dots, \ell_n)$, such that translating each C_i by ℓ_i results in n disjoint curves. We have a natural morphism from U to the Hilbert scheme of X. Explicitly, let Z_i be the product $X^{i-1} \times C_i \times X^{n-i}$, for $1 \leq i \leq n$. Then F is the pullback of the ideal of the union $\cup_{i=1}^n Z_i$ via trhe diagonal embedding of X in X^n. $\mathrm{Pic}^0(C)^n$ acts on X^n introducing the desired map from the set U to the Hilbert scheme of X. The symmetric group $\mathfrak{S}_n$ acts freely on U as follows. If $(t_1, t_2, \dots, t_n) \in \mathrm{Pic}^1(C)^n$ translates $C \times \cdots \times C$ to $C_1 \times C_2 \times \cdots \times C_n$, then U is invariant with respect to the action on $\mathrm{Pic}^0(X)^n$ which is conjugated by translation by $(t_1, t_2, \dots, t_n)$ to the permutation action on $\mathrm{Pic}^1(X)^n$ and the action on U is fixed point free. The connected component of F in the moduli space of simple sheaves on X contains a smooth subscheme isomorphic to $(U/\mathfrak{S}_n) \times \mathrm{Pic}^0(X)$ obtained by translating the C_i's and tensoring F by line bundles. Hence, we have a canonical injective homomorphism

$$H^0(X, TX)^n \oplus H^1(X, \mathcal{O}_X) \to \mathrm{Ext}^1(F, F) \tag{8.2.1}$$

and $\dim \mathrm{Ext}^1(F, F) \geq 3n + 3$.

Lemma 8.2.2. *The homomorphism (8.2.1) is an isomorphism. Consequently,* $\dim Ext^1(F, F) = 3n + 3$.

Proof. It suffices to prove the inequality $\dim \mathrm{Ext}^1(F, F) \leq 3n + 3$. Set $\mathcal{I}_k := \mathcal{I}_{\cup_{i=1}^k C_i}$. We have the short exact sequence

$$0 \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \mathcal{O}_{C_n} \to 0 \tag{8.2.2}$$

and the long exact

$$0 \to \mathrm{Hom}(\mathcal{I}_n, \mathcal{I}_n) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{I}_n, \mathcal{I}_{n-1}) \xrightarrow{0} \mathrm{Hom}(\mathcal{I}_n, \mathcal{O}_{C_n}) \to \mathrm{Ext}^1(\mathcal{I}_n, \mathcal{I}_n) \to \mathrm{Ext}^1(\mathcal{I}_n, \mathcal{I}_{n-1})$$

We have the isomorphism $\mathrm{Hom}(\mathcal{I}_n, \mathcal{O}_{C_n}) \cong H^0(C_n, N_{C_n/X})$. The quotient of $H^0(C_n, N_{C_n/X})$ by $H^0(C_n, TX_{|C_n})$ is the kernel of the differential of the Torelli map $H^1(C_n, TC_n) \to H^1(C_n, TX_{|C_n}) \cong H^{0,1}(C_n) \otimes H^{0,1}(C_n)$, and the latter is injective for our non-hyperelliptic curve C_n, by Noether's theorem. Hence, $H^0(C_n, N_{C_n/X})$ is 3-dimensional. It remains to prove the inequality $\dim \mathrm{Ext}^1(\mathcal{I}_n, \mathcal{I}_{n-1}) \leq 3n$. We will prove by induction on n the equality $\dim \mathrm{Ext}^1(\mathcal{I}_n, \mathcal{I}_{n-1}) = 3n$.

When $n=1$, $\mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1})\cong H^2(\mathcal{I}_1)^*\cong H^2(X,\mathcal{I}_{C_1})^*$ is 3-dimensional. Indeed, we have more generally the short exact sequence

$$0\rightarrow \frac{\oplus_{i=1}^n H^1(\mathcal{O}_{C_i})}{H^1(\mathcal{O}_X)}\rightarrow H^2(\mathcal{I}_n)\rightarrow H^2(\mathcal{O}_X)\rightarrow 0,$$

obtained from the long exact sequence of cohomology associated to the short exact $0\rightarrow\mathcal{I}_n\rightarrow\mathcal{O}_X\rightarrow\oplus_{i=1}^n\mathcal{O}_{C_i}\rightarrow 0$ using the injectivity of $H^1(\mathcal{O}_X)\rightarrow\oplus_{i=1}^n H^1(C_i,\mathcal{O}_{C_i})$.

Assume that $n\geq 2$ and $\dim\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{I}_{n-2})=3(n-1)$. Then $\dim\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{I}_{n-1})=3n$, as shown above. Consider long exact sequence obtained from the short exact sequence (8.2.2)

$$\begin{array}{rll} 0\rightarrow & \mathrm{Hom}(\mathcal{I}_{n-1},\mathcal{I}_{n-1}) & \stackrel{\cong}{\rightarrow}\mathrm{Hom}(\mathcal{I}_{n-1},\mathcal{O}_{C_n})\\ \stackrel{0}{\rightarrow}\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{I}_n)\rightarrow & \mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{I}_{n-1}) & \rightarrow\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{O}_{C_n})\\ \stackrel{\xi}{\rightarrow}\mathrm{Ext}^2(\mathcal{I}_{n-1},\mathcal{I}_n)\rightarrow & \mathrm{Ext}^2(\mathcal{I}_{n-1},\mathcal{I}_{n-1}) & \rightarrow 0. \end{array}$$

The sheaves $\mathcal{E}xt^i(\mathcal{I}_{n-1},\mathcal{O}_{C_n})$ vanish, for $i>0$. Hence, the local to global spectral sequence computing $\mathrm{Ext}^i(\mathcal{I}_{n-1},\mathcal{O}_{C_n})$ degenerates at the E_2 term. We get the vanishing of $\mathrm{Ext}^i(\mathcal{I}_{n-1},\mathcal{O}_{C_n})$, for $i>1$ and the isomorphism $\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{O}_{C_n})\cong H^1(\mathcal{O}_{C_n})$ and the latter is 3-dimensional. Furthermore, the composition

$$H^1(\mathcal{O}_X)\cong H^1(\mathcal{E}nd(\mathcal{I}_{n-1},\mathcal{I}_{n-1}))\rightarrow\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{I}_{n-1})\rightarrow\mathrm{Ext}^1(\mathcal{I}_{n-1},\mathcal{O}_{C_n})\cong H^1(\mathcal{O}_{C_n})$$

is surjective. Hence, the connecting homomorphism ξ vanishes and

$$\dim\mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1})=\dim\mathrm{Ext}^2(\mathcal{I}_{n-1},\mathcal{I}_n)=\dim\mathrm{Ext}^2(\mathcal{I}_{n-1},\mathcal{I}_{n-1})=3n,$$

where the first equality is by Serre's duality, the second by the vanishing of ξ, and the third was established above via the induction hypothesis. $\square$

Let $\Delta_X\subset X\times X$ be the diagonal. A class in the Hochschild cohomology $HH^i(X):=\mathrm{Hom}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}[i])$ corresponds to a natural transformation from the identity endofunctor of $D^b(X)$ to its shift by i. The evaluation homomorphism

$$ev_F: HH^*(X)\rightarrow\mathrm{Ext}^*(F,F) \tag{8.2.3}$$

is a graded algebra homomorphism. Denote by $ev_F^i: HH^i(X)\rightarrow\mathrm{Ext}^i(F,F)$ the restriction of ev_F to $HH^i(X)$.

The second Hochschild cohomology $HH^2(X)$ parametrizes first order deformations of $D^b(X)$. Let

$$ob_F: HH^2(X)\rightarrow\mathrm{Ext}^2(F,F)$$

be the homomorphism ev_F^2. The kernel of ob_F parametrizes those deformations along which F deforms to first order, by [T]. As above, we set $F:=\mathcal{I}_{\cup_{i=1}^n C_i}$, where $n\geq 1$. Denote by $ob_F: HT^2(X)\rightarrow\mathrm{Ext}^2(F,F)$ also the composition of ob_F above with the HKR isomorphism $HT^2(X):=H^2(\mathcal{O}_X)\oplus H^1(TX)\oplus H^0(\wedge^2TX)\cong HH^2(X)$. Then ob_F is given by contraction with the exponential Atiyah class $\exp(at_E)$, by [Hua, Th. A].

Lemma 8.2.3. $\mathrm{rank}(ob_F)\geq 6$.

Proof. Consider the contraction homomorphism

$$H^2(\mathcal{O}_X) \oplus H^1(TX) \oplus H^0(\wedge^2 TX) \stackrel{\rfloor ch(F)}{\longrightarrow} H^2(\mathcal{O}_X) \oplus H^3(\Omega^1_X).$$

It restricts to the first summand as an isomorphism onto the first summand of the co-domain, as $ch_0(F) = 1$, and to the second summand as an isomorphism onto the second summand of the co-domain, as $ch_2(F) = \frac{-n}{2}\Theta^2$. Hence, the above homomorphism is sujective and so its kernel has co-dimension 6. The kernel of ob_F is contained in the kernel of the above homomorphism, by [Hua, Theorem B], hence ob_F has rank ≥ 6. $\square$

Diagram (7.1.2) extends to the commutative diagram

$$\begin{tikzcd} HT^2(M) \arrow[rr, "ob_E"] \arrow[dr, "\rfloor ch(E)"'] & & \mathrm{Ext}^2(E,E) \arrow[dl, "\sigma"] \\ & \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M) & \end{tikzcd} \tag{8.2.4}$$

by [BF2, Prop. 6.2.1 and Cor. 6.3.2].

Lemma 8.2.4. *If the kernels of ob_E and $\rfloor ch(E)$ in $HT^2(M)$ are equal and ob_E is surjective, then E is semiregular.*

Proof. The hypotheses imply that there exists a unique injective map $\sigma' : \mathrm{Ext}^2(E,E) \to \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M)$, such that $\rfloor ch(E) = \sigma' \circ ob_E$. The equality $\sigma = \sigma'$ follows from the commutativity of Diagram (8.2.4). $\square$

Remark 8.2.5. If we drop the assumption that ob_E is surjective in Lemma 8.2.4 we still conclude that the semiregularity map restricts to the image of the obstruction map as an injective map.

Remark 8.2.6. Note that the hypotheses of Lemma 8.2.4 are invariant under derived equivalences. If $\Phi : D^b(M) \to D^b(M')$ is an equivalence of derived categories, E satisfies the hypotheses of Lemma 8.2.4, and $\Phi(E)$ is represented by a coherent sheaf E', then E' satisfies the hypotheses of Lemma 8.2.4 and is thus semiregular as well. The space $\prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M)$ in the above diagrams is the graded summand $H\Omega_{-2}(M)$ of the Hochschild homology $HH_*(M)$. The equivalence Φ induces isomorphisms $\Phi : \mathrm{Ext}^2(E,E) \to \mathrm{Ext}^2(E',E')$ and $\Phi_* : H\Omega_{-2}(M) \to H\Omega_{-2}(M')$. The hypotheses of Lemma 8.2.4 imply that the semiregularity map σ' of E' is the conjugate of the semiregularity map σ of E, $\sigma' \circ \Phi = \Phi_* \circ \sigma$. It is natural to expect that the latter equality holds, more generally, without the hypotheses of Lemma 8.2.4.

Lemma 8.2.7. (1) *The algebra* $\mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ *is generated by* $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$*.*

(2) *The homomorphism $ev_{\mathcal{I}_{C_j}} : HT^*(X) \to \mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is surjective and its kernel is the annihilator* $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ *of the Chern character $1 - \frac{1}{2}\Theta^2 + 2[pt]$ of*

$\mathcal{I}_{C_j}$.

$$0 \to \mathrm{ann}(ch(\mathcal{I}_{C_j})) \cap HT^2(X) \to \begin{array}{c} H^2(\mathcal{O}_X) \\ \oplus \\ H^1(TX) \\ \oplus \\ H^0(\wedge^2 TX) \end{array} \overset{\left(\begin{array}{ccc} 1 & 0 & -\Theta^2/2 \\ 0 & -\Theta^2/2 & 2[pt] \end{array}\right)}{\longrightarrow} \begin{array}{c} H^2(\mathcal{O}_X) \\ \oplus \\ H^3(\Omega^1_X) \end{array} \to 0.$$

(3) *The sheaf* $\mathcal{I}_{C_j}$ *is semiregular.*

Proof. (1) This is the case $n = 1$. In this case $\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is 6-dimensional, by Lemma 8.2.2, and so $ob_{\mathcal{I}_{C_j}}$ is surjective, by Lemma 8.2.3. Now $HT^*(X)$ is generated by $HT^1(X)$ and $ev_{\mathcal{I}_{C_j}}$ is an algebra homomorphism. Hence, the surjectivity of $ev^2_{\mathcal{I}_{C_j}} = ob_{\mathcal{I}_{C_j}}$ implies that the Yoneda product $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \otimes \mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \to \mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is surjective. The surjectivity of $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \otimes \mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \to \mathrm{Ext}^3(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ follows from Serre's duality.

(2) The homomorphism $ev^1_{\mathcal{I}_{C_j}} : HT^1(X) \to \mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is an isomorphism, by Lemma 8.2.2. Hence, the homomorphism $ev_{\mathcal{I}_{C_j}}$ is surjective, by part (1). The inclusion $\ker(ev_{\mathcal{I}_{C_j}}) \subset \mathrm{ann}(ch(\mathcal{I}_{C_j}))$ follows from [Hua, Theorem B]. Both ideals are graded (homogeneous). Indeed, the homomorphism $ev_{\mathcal{I}_{C_j}}$ is graded, by definition, and contraction with $ch(\mathcal{I}_{C_j})$ maps $HT^k(X)$ to $\oplus_{q-p=k} H^{p,q}(X)$. The graded summands of $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ in $HT^0(X)$ and $HT^1(X)$ vanish. The equality of $\ker(ev^2_{\mathcal{I}_{C_j}})$ and the graded summand of $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ in $HT^2(X)$ follows from the proof of Lemma 8.2.3. The graded summands of both ideals in $HT^3(X)$ have co-dimension 1, since $\mathrm{Ext}^3(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ and $H^3(\mathcal{O}_X)$ are both one-dimensional, and the summand $H^3(\mathcal{O}_X)$ of $HT^3(X)$ surjects onto both. Hence the inclusion $\ker(ev_{\mathcal{I}_{C_j}}) \subset \mathrm{ann}(ch(\mathcal{I}_{C_j}))$ implies the equality of the graded summands in $HT^3(X)$ of both ideals. $HT^k(X)$ is contained in both ideals for $k > 3$ for degree reasons.

(3) Follows immediately from Part (2) and Lemma 8.2.4. □

The proof of Lemma 8.2.2 identifies the i-th direct summand in the domain of (8.2.1) with $H^0(C_i, N_{C_i/X})$, $1 \leq i \leq n$. Denote this direct summand by $\tilde{E}^1_i$ and set $\tilde{E}^1_0 := H^1(X, \mathcal{O}_X)$, so that the domain of (8.2.1) is $\oplus_{i=0}^n \tilde{E}^1_i$. Note that each $\tilde{E}^1_i$ is 3-dimensional. We denote by E^1_i the image of $\tilde{E}^1_i$ via the isomorphism (8.2.1). The Yoneda product $\mathrm{Ext}^1(F,F) \otimes \mathrm{Ext}^2(F,F) \to \mathrm{Ext}^3(F,F)$ is a perfect pairing. Let E^2_i, $1 \leq i \leq n$, be the subspace of $\mathrm{Ext}^2(F,F)$ annihilating $E^1_0 \oplus \oplus_{j=1, j\neq i}^n E^1_i$. Let E^2_0 be the image of $H^2(\mathcal{O}_X)$ in $\mathrm{Ext}^2(F,F)$. Then E^2_i is 3-dimensional, for $0 \leq i \leq n$. The Yoneda product restricts to $E^1_0 \otimes E^2_0 \to \mathrm{Ext}^3(F,F)$ as a perfect pairing. Indeed, the algebra homomorphism

$$\iota : \mathrm{Ext}^*(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Ext}^*(F,F)$$

is injective, since it composes with the trace linear homomorphism $tr : \mathrm{Ext}^*(F,F) \to \mathrm{Ext}^*(\mathcal{O}_X, \mathcal{O}_X)$ to the identity of $\mathrm{Ext}^*(\mathcal{O}_X, \mathcal{O}_X)$, by [HL, 10.1.3] and the equality $\mathrm{rank}(F) = 1$. We get the direct sum decomposition

$$\mathrm{Ext}^2(F,F) = \oplus_{i=0}^n E^2_i.$$

When considering different ideal sheaves $\mathcal{I}_{\cup_{i=1}^n C_i}$ we will denote E^i_j by $E^i_j(\mathcal{I}_{\cup_{i=1}^n C_i})$. We have the isomorphism $F := \mathcal{I}_{\cup_{i=1}^n C_i} \cong \otimes_{i=1}^n \mathcal{I}_{C_i}$, hence the functor of tensoring with $\otimes_{i=1, i\neq j}^n \mathcal{I}_{C_i}$ induces an algebra homomorphism

$$e_j : \mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \to \mathrm{Ext}^*(F, F).$$

The ring structure of the Yoneda algebra $\mathrm{Ext}^*(F, F)$ is determined by the following Lemma and Lemma 8.2.7.

Lemma 8.2.8. (1) *The Yoneda product maps $E^1_i \otimes E^1_j$ to zero in $\mathrm{Ext}^2(F, F)$, if $i \neq j$ and $1 \leq i, j \leq n$.*
(2) *The Yoneda product $\mathrm{Ext}^1(F, F) \otimes \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ is anti-symmetric.*
(3) *The Yoneda product maps $(E^1_0 \oplus E^1_i) \otimes (E^1_0 \oplus E^1_i)$ surjectively onto $E^2_0 \oplus E^2_i$, for all $1 \leq i \leq n$. In particular, the algebra $\mathrm{Ext}^*(F, F)$ is generated by $\mathrm{Ext}^1(F, F)$.*
(4) *The image of e_j is $\mathrm{Hom}(F, F) \oplus (E^1_0 \oplus E^1_j) \oplus (E^2_0 \oplus E^2_j) \oplus \mathrm{Ext}^3(F, F)$.*
(5) *The homomorphism e_j maps $E^d_1(\mathcal{I}_{C_j})$ isomorphically onto $E^d_j(F)$, for $d = 1, 2$.*

Proof. (1) Let ξ_i be a section of $H^0(N_{C_i/X})$ and $\tilde{\xi}_i$ the corresponding class in E^1_i. Set $R := \mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2\rangle$. The product $\tilde{\xi}_i \circ \tilde{\xi}_j$ vanishes, if and only if there exists a deformation of $\mathcal{I}_{\cup_{k=1}^n C_k}$ by an ideal over $X \times \mathrm{Spec}(R)$, which restricts to the first order deformation along ξ_i over $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1^2, \epsilon_2, \rangle)$ and to the first order deformation along ξ_j over $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1, \epsilon_2^2\rangle)$, by [Ar, Sec. 2]. Assume that $i \neq j$. Let $\mathcal{F}$ be the ideal sheaf over $X \times \mathrm{Spec}(R)$ consisting of elements locally of the form $f_0 + f_1\epsilon_1 + f_2\epsilon_2$, where $f_0 \in F := \mathcal{I}_{\cup_{k=1}^n C_k}$, f_1 belongs to $\mathcal{I}_{\cup_{k=1, k\neq i}^n C_k}$, f_2 belongs to $\mathcal{I}_{\cup_{k=1, k\neq j}^n C_k}$,

$$(f_1)_{|C_i} + df_0(\xi_i) = 0, \text{ and } (f_2)_{|C_j} + df_0(\xi_j) = 0.$$

One easily checks that $\mathcal{F}$ is indeed an ideal and $\mathcal{F}$ clearly restricts to the ideals of the two desired first order deformations.

(2) The anti-symmetry would follow from that of $(E^1_0 \oplus E^1_i) \otimes (E^1_0 \oplus E^1_i) \to \mathrm{Ext}^2(F, F)$, by Part (1). Now, $E^1_0 \oplus E^1_i$ is equal to $e_i(\mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}))$, e_i is an algebra homomorphism, and $\mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}) \otimes \mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}) \to \mathrm{Ext}^2(\mathcal{I}_{C_i}, \mathcal{I}_{C_i})$ is anti-symmetric, by the anti-symmetry of the product $HT^1(X) \otimes HT^1(X) \to HT^2(X)$ and the surjectivity of $ev_{\mathcal{I}_{C_i}}$ in Lemma 8.2.7.

(3) We prove first that the image of $(E^1_0 \oplus E^1_i) \otimes (E^1_0 \oplus E^1_i)$ is contained in $E^2_0 \oplus E^2_i$. It suffices to prove that the image of $E^1_i \otimes \mathrm{Ext}^1(F, F)$ is contained in $E^2_0 \oplus E^2_i$. This follows from part (1) of the lemma as $E^1_i \otimes \mathrm{Ext}^1(F, F)$ annihilates E^1_j, for all $j \neq i$ (here we use also the anti-symmetry property in part (2)). Surjectivity would follow from the proof of part (4) and Lemma 8.2.7.

(4) We know that e_j maps $\mathrm{Ext}^k(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ onto $\mathrm{Ext}^k(F, F)$, for $k = 0$ and $k = 3$. We also know that $e_j(\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})) = E^1_0 \oplus E^1_j$. It remains to show that $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})) = E^2_0 \oplus E^2_j$. Clearly, $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}))$ is contained in the image of $(E^1_0 \oplus E^1_j) \otimes (E^1_0 \oplus E^1_j)$ as e_j is an algebra homomorphism. Hence, $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}))$ is contained in $E^2_0 \oplus E^2_j$, by part (3). Now the fact that the restriction of e_j is injective on $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ implies that it is also injective on $\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$, since the pairing induced by the Yoneda product of their images in $\mathrm{Ext}^*(F, F)$ pulls back to the perfect pairing of the Yoneda product

in $\mathrm{Ext}^*(\mathcal{I}_{C_j},\mathcal{I}_{C_j})$, as e_j induces an isomorphism of $\mathrm{Ext}^3(\mathcal{I}_{C_j},\mathcal{I}_{C_j})$ with $\mathrm{Ext}^3(F,F)$. The equality $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j},\mathcal{I}_{C_j}))=E_0^2\oplus E_j^2$ follows for dimension reasons.

(5) The statement is clear for $d=1$. For $d=2$ it follows from part (4) and the fact that e_j is an $H^*(\mathcal{O}_X)$-algebra homomorphism. □

Proposition 8.2.9. $\mathrm{rank}(ob_F)=6$.

Proof. We already know that $\mathrm{rank}(ob_F)\geq 6$, by Lemma 8.2.3. It remains to prove that $\mathrm{rank}(ob_F)\leq 6$. If $n=1$, then $\dim \mathrm{Ext}^2(F,F)=6$, by Lemma 8.2.2, and so $\mathrm{rank}(ob_F)=6$.

Denote by $ev_F: HT^*(X)\rightarrow \mathrm{Ext}^*(F,F)$ also the composition of (8.2.3) with the HKR isomorphism $HT^*(X)\cong HH^*(X)$. The HKR isomorphism is an $H^*(\mathcal{O}_X)$-algebra isomorphism, since the Todd class of X vanishes [CBR]. Hence, the latter ev_F is an $H^*(\mathcal{O}_X)$-algebra homomorphism. An element $\xi\in HT^1(X)$ decomposes uniquely as the sum $\xi'+\xi''$ with $\xi'\in H^1(\mathcal{O}_X)$ and $\xi''\in H^0(TX)$. We have the following equalities:

$$
\begin{aligned}
ev_F(\xi') &= e_j(ev_{\mathcal{I}_{C_j}}(\xi')),\\
ev_F(\xi'') &= \sum_{j=1}^n e_j(ev_{\mathcal{I}_{C_j}}(\xi'')),
\end{aligned}
\tag{8.2.5}
$$

by Lemma 8.2.2. Note that $e_j\circ ev_{\mathcal{I}_{C_j}}$ is a composition of $H^*(\mathcal{O}_X)$-algebra homomorphisms. If $j\neq k$, then $e_j(ev_{\mathcal{I}_{C_j}}(\xi_1''))e_k(ev_{\mathcal{I}_{C_k}}(\xi_2''))=0$, for every two elements $\xi_1,\xi_2\in HT^1(X)$, by Lemma 8.2.8. We get

$$
\begin{aligned}
ev_F(\xi_1\xi_2) &= \left[ev_F(\xi_1')+\sum_{j=1}^n e_j(ev_{\mathcal{I}_{C_j}}(\xi_1''))\right]\left[ev_F(\xi_2')+\sum_{j=1}^n e_j(ev_{\mathcal{I}_{C_j}}(\xi_2''))\right]\\
&= ev_F(\xi_1'\xi_2')+\sum_{j=1}^n e_j(ev_{\mathcal{I}_{C_j}}(\xi_1'\xi_2''+\xi_1''\xi_2'+\xi_1''\xi_2''))\\
&= \sum_{j=1}^n e_j\left(ev_{\mathcal{I}_{C_j}}\left(\frac{1}{n}\xi_1'\xi_2'+\xi_1'\xi_2''+\xi_1''\xi_2'+\xi_1''\xi_2''\right)\right).
\end{aligned}
$$

The algebra $HT^*(X)$ is generated by $HT^1(X)$. The element $\xi_1'\xi_2'$ belongs to $H^2(\mathcal{O}_X)$, while the element $\xi_1'\xi_2''+\xi_1''\xi_2'+\xi_1''\xi_2''$ belongs to $H^1(TX)\oplus H^0(\wedge^2TX)$. Thus, the two equations (8.2.5) hold also for $\xi\in HT^2(X)$ under the decomposition $\xi=\xi'+\xi''$, with $\xi'\in H^2(\mathcal{O}_X)$ and $\xi''\in H^1(TX)\oplus H^0(\wedge^2TX)$.

Let $\tau_{ij}:D^b(X)\rightarrow D^b(X)$ be the autoequivalence induced by the translation automorphism mapping C_j to C_i. This autoequivalence acts trivially on $HT^*(X)$ and $ev_{\mathcal{I}_{C_i}}=\tau_{ij}\circ ev_{\mathcal{I}_{C_j}}$. Furthermore, e_j is injective, by Lemma 8.2.8. Hence, the kernel of the composition $e_j\circ ev_{\mathcal{I}_{C_j}}: HT^*(X)\rightarrow \mathrm{Ext}^*(F,F)$ is independent of j. Let

$$\gamma_n: HT^2(X)\rightarrow HT^2(X)$$

be the automorphism multiplying the direct summand $H^2(\mathcal{O}_X)$ by n and acting as the identity on $H^1(TX)\oplus H^0(\wedge^2TX)$. We see that γ_n maps the kernel of $e_j\circ ob_{\mathcal{I}_{C_j}}$ into that of ob_F. Now e_j is injective, by Lemma 8.2.8, and $ev^2_{\mathcal{I}_{C_j}} = ob_{\mathcal{I}_{C_j}}$ has rank 6, by the case $n=1$. Hence, $\mathrm{rank}(ob_F) \leq \mathrm{rank}(ob_{\mathcal{I}_{C_j}}) = 6$. ☐

Lemma 8.2.10. *Let $\Phi : D^b(A) \to D^b(B)$ be an equivalence of derived categories of two abelian varieties and F an object of $D^b(A)$. Assume that the kernel of $ob_F : HT^2(A) \to \mathrm{Hom}(F, F[2])$ is equal to the subspace annihilating $ch(F)$. Then the kernel of $ob_{\Phi(F)}$ is equal to the subspace annihilating $ch(\Phi(F))$.*

Proof. We have the commutative diagram

$$\begin{tikzcd}
H^*(A,\mathbb{C}) \arrow[d, "\Phi^H"'] & HT^2(A) \arrow[l, "ch(F)"'] \arrow[r, "\exp(at_F)"] \arrow[d, "\Phi^{HT}"] & \mathrm{Hom}(F,F[2]) \arrow[d, "\Phi"] \\
H^*(B,\mathbb{C}) & HT^2(B) \arrow[l, "ch(E)"] \arrow[r, "\exp(at_E)"'] & \mathrm{Hom}(E,E[2]),
\end{tikzcd}$$

the left square by [CBR], and the right square by [Hua, Theorem A]. The vertical arrows are isomorphisms and the kernels of the two horizontal arrows in the top row are equal. Hence, the same is true for the bottom row. ☐

Remark 8.2.11. The above Lemma holds for more general projective varieties, replacing $ch(F)$ by the Mukai vector $v(F) := ch(F)td(X)^{\frac{1}{2}}$ and factoring the action of $HT^*(X)$ on its module $H^*(X,\mathbb{C})$ by the Duflo operator $D : HT^*(X) \to HT^*(X)$, given by $D(\alpha) = td(X)^{\frac{1}{2}} \rfloor \alpha$. For abelian varieties $td(X)^{\frac{1}{2}} = 1$.

Corollary 8.2.12. *Set $F := \mathcal{I}_{\cup_{i=1}^n C_i}(\Theta)$. The kernel of ob_F is equal to the kernel of the homomorphism $(\bullet)\rfloor ch(F) : HT^2(X) \to H^*(X,\mathbb{C})$ of contraction with the Chern character of F.*

Proof. It suffices to prove the statement for $F' := \mathcal{I}_{\cup_{i=1}^n C_i}$, by Lemma 8.2.10, as F is the image of F' by the autoequivalence of tensorization by Θ. Now, $ch(\mathcal{I}_{\cup_{i=1}^n C_i}) = 1 - \frac{n}{2}\Theta^2 + 2n[pt]$. The kernel of $ob_{F'}$ is contained in the kernel K of $\rfloor ch(F')$, by [Hua, Theorem B]. The kernel of $ob_{F'}$ is 9 dimensional, by Proposition 8.2.9. It suffices to show that $\dim(K) \leq 9$. Assume that $\dim(K) > 9$. Then the intersection $K \cap [H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)]$ would be non-trivial. We claim that $K \cap [H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)] = (0)$. Indeed, contraction with $ch(F')$ induces the homomorphism

$$H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX) \to H^2(\mathcal{O}_X) \oplus H^3(\Omega^1_X)$$

with upper triangular matrix $\begin{pmatrix} 1 & -\frac{n}{2}\Theta^2 \\ 0 & 2n[pt] \end{pmatrix}$, which is invertible ☐

8.3. **Secant$^{\boxtimes 2}$-sheaves over $X \times \hat{X}$ with a 9-dimensional space of unobstructed commutative-gerby deformations.** Let π_i, $i = 1, 2$, be the projections from $X \times X$ to X. Denote by at_F the Atiyah class of F. The Atiyah class $at_{\pi_1^* F} \in \mathrm{Ext}^1(\pi_1^* F, (\pi_1^* F) \otimes \Omega^1_{X\times X})$ of $\pi_1^* F$ is equal to the pushforward of $\pi_1^* at_F \in \mathrm{Ext}^1(\pi_1^* F, \pi_1^*(F \otimes \Omega^1_X))$ via the inclusion of $\pi_1^*\Omega^1_X$ as a direct summand in $\Omega^1_{X\times X}$ [BF1, Prop. 3.14]. Let F_1 and F_2 be the sheaves in Theorem 1.4.1. The Atiyah class of $\pi_1^* F_1 \otimes \pi_2^* F_2$ satisfied

$$at_{\pi_1^* F_1 \otimes \pi_2^* F_2} = at_{\pi_1^* F_1} \otimes 1 + 1 \otimes at_{\pi_2^* F_2}. \tag{8.3.1}$$

The Künneth decomposition of $\mathrm{Ext}^2(\pi_1^*F_1\otimes\pi_2^*F_2,\pi_1^*F_1\otimes\pi_2^*F_2)$ is the direct sum $[\mathrm{Ext}^2(F_1,F_1)\otimes\mathrm{Ext}^0(F_2,F_2)]\oplus[\mathrm{Ext}^0(F_1,F_1)\otimes\mathrm{Ext}^2(F_2,F_2)]\oplus[\mathrm{Ext}^1(F_1,F_1)\otimes\mathrm{Ext}^1(F_2,F_2)]$. We have the direct sum decomposition of $HT^2(X\times X)$

$$\begin{aligned} HT^2(X\times X) &= \pi_1^*HT^2(X)\otimes\pi_2^*HT^0(X)\oplus \\ & \quad \pi_1^*HT^0(X)\otimes\pi_2^*HT^2(X)\oplus \\ & \quad \pi_1^*HT^1(X)\otimes\pi_2^*HT^1(X) \end{aligned} \tag{8.3.2}$$

Lemma 8.2.2 implies that $\exp(at_F): H^1(\mathcal{O}_X)\oplus H^0(TX)\to\mathrm{Ext}^1(F,F)$ is injective. The obstruction map $ob_F: HT^2(X)\to\mathrm{Ext}^2(F,F)$ is the restriction to $HT^2(X)$ of the algebra homomorphism

$$(\bullet)\rfloor\exp(at_F): HT^*(X)\to\mathrm{Ext}^*(F,F)$$

(see [Hua, Theorem A]). We see that the obstruction map $ob_{\pi_1^*F_1\otimes\pi_2^*F_2}$ maps the summand in the i-th row above into the i-th summand of $\mathrm{Ext}^2(\pi_1^*F_1\otimes\pi_2^*F_2,\pi_1^*F_1\otimes\pi_2^*F_2)$ in the decomposition displayed above and $ob_{\pi_1^*F_1\otimes\pi_2^*F_2}$ restricts as an injective homomorphism to the third summand. We get

$$\ker(ob_{\pi_1^*F_1\otimes\pi_2^*F_2})=[\pi_1^*\ker(ob_{F_1})\otimes\pi_2^*H^0(\mathcal{O}_X)]\oplus[\pi_1^*H^0(\mathcal{O}_X)\otimes\pi_2^*\ker(ob_{F_2})]. \tag{8.3.3}$$

Let $\Phi: D^b(X\times X)\to D^b(X\times\hat{X})$ be Orlov's derived equivalence (6.1.2) and set

$$E:=\Phi(\pi_1^*F_1\otimes\pi_2^*F_2). \tag{8.3.4}$$

Lemma 8.3.1. (1) *The kernel of $ob_{\pi_1^*F_1\otimes\pi_2^*F_2}$ is equal to the subspace of $HT^2(X\times X)$ annihilating $ch(\pi_1^*F_1\otimes\pi_2^*F_2)$.*

(2) *The kernel of ob_E is equal to the subspace of $HT^2(X\times\hat{X})$ annihilating $ch(E)$.*

Proof. (1) Let Z be the subspace of $HT^2(X\times X)$ annihilating $ch(\pi_1^*F_1\otimes\pi_2^*F_2)$. The inclusion $\ker(ob_{\pi_1^*F_1\otimes\pi_2^*F_2})\subset Z$ follows from [Hua, Theorem B]. Equation (8.3.3) implies that $\ker(ob_{\pi_1^*F_1\otimes\pi_2^*F_2})$ is contained in

$$[\pi_1^*HT^2(X)\otimes\pi_2^*HT^0(X)]\oplus[\pi_1^*HT^0(X)\otimes\pi_2^*HT^2(X)]. \tag{8.3.5}$$

We have seen that the kernel of ob_{F_i} is equal to the subspace of $HT^2(X)$ annihilating $ch(F_i)$ under the action of $HT^*(X)$ on its module $H^*(X,\mathbb{C})$ (Corollary 8.2.12). In order to prove the inclusion $Z\subset\ker(ob_{\pi_1^*F_1\otimes\pi_2^*F_2})$ it suffices to prove that Z is contained in (8.3.5), by the decomposition (8.3.1) of the Atiyah class of $\pi_1^*F_1\otimes\pi_2^*F_2$. The homomorphism

$$(\pi_1^*ch(F_1)\cup\pi_2^*ch(F_2))\rfloor(\bullet): HT^2(X\times X)\to H^*(X\times X,\mathbb{C})$$

maps the third summand $\pi_1^*HT^1(X)\otimes\pi_2^*HT^1(X)$ in the decomposition (8.3.2) to a subspace of $H^*(X\times X,\mathbb{C})$ intersecting trivially the sum of the images of the other two summands. Indeed, the first summand in (8.3.2) is mapped into $\pi_1^*H^*(X,\mathbb{C})\otimes\pi_2^*ch(F_2)$, the second into $\pi_1^*ch(F_1)\otimes\pi_2^*H^*(X,\mathbb{C})$ and every element in the direct sum of the latter two is the sum of classes of the form $\pi_1^*\alpha\cup\pi_2^*\beta$, where either α or β is a Hodge class. On the other hand, the image of $\pi_1^*HT^1(X)\otimes\pi_2^*HT^1(X)$ is contained in the subspace

$$\pi_1^*[\oplus_{q-p=1}H^{p,q}(X)]\otimes\pi_2^*[\oplus_{q-p=1}H^{p,q}(X)].$$

In order to prove that Z is contained in (8.3.5) it suffices to prove that the homomorphism $ch(F_i)\rfloor(\bullet) : HT^1(X) \to H^*(X,\mathbb{C})$ is injective. Indeed, multiplication by $1 = \mathrm{rank}(F_i)$ induces an injective homomorphism from the subspace $H^1(\mathcal{O}_X)$ of $HT^1(X)$ to the subspace $H^1(\mathcal{O}_X)$ of $H^1(X,\mathbb{C})$ and contraction with $ch_2(F_i) = -\frac{n}{2}\Theta^2$ induces an injective homomorphism from $H^0(TX)$ to $H^{1,2}(X)$.

(2) Apply Lemma 8.2.10 with $A = X \times X$, $B = X \times \hat{X}$, and $F = \pi_1^*F_1 \otimes \pi_2^*F_2$. $\square$

8.4. **Orlov's isomorphism $\Phi^{HT} : HT^2(X \times X) \to HT^2(X \times \hat{X})$ maps diagonal deformations to commutative-gergy ones.** The algebra $HT^*(X)$ acts on its module $H^*(X,\mathbb{C}) := \oplus H^{p,q}(X)$ and embeds in $\mathrm{End}(H^*(X,\mathbb{C}))$. Given $\alpha \in HT^*(X)$ denote by $e_\alpha \in \mathrm{End}(H^*(X,\mathbb{C}))$ the corresponding endomorphism. Let τ be the involution in (1.2.3). If α is an element of $H^i(\wedge^j TX)$ and x is a class in $H^k(X,\mathbb{C})$, then $e_\alpha(x)$ belongs to $H^{k+i-j}(X,\mathbb{C})$. Set $t := i - j$. We have

$$(\tau \circ e_\alpha \circ \tau)(x) = (-1)^{\frac{(k+t)(k+t-1)}{2}}(-1)^{\frac{k(k-1)}{2}}e_\alpha(x) = (-1)^{kt+\frac{t(t-1)}{2}}e_\alpha(x).$$

In particular, for k even, we have $(\tau \circ e_\alpha \circ \tau)(x) = (-1)^{\frac{t(t-1)}{2}}e_\alpha(x)$. In particular,

$$(\tau \circ e_\alpha \circ \tau)(ch(F)) = (-1)^{\frac{t(t-1)}{2}}e_\alpha(ch(F))$$

Let $(\bullet)^* : HT^*(X) \to HT^*(X)$ act on $H^i(\wedge^j TX)$ by multiplication by $(-1)^{\frac{(i-j)(i-j-1)}{2}}$. We get

$$(\tau \circ e_\alpha \circ \tau)(ch(F)) = e_{\alpha^*}(ch(F)). \tag{8.4.1}$$

Recall that $\tau(ch(F)) = ch(F^\vee)$. In particular, e_{α^*} annihilates $ch(F^\vee)$, if and only if e_α annihilates $ch(F)$. For $(\alpha,\beta,\gamma) \in HT^2(X) = H^2(\mathcal{O}_X) \oplus H^1(TX) \oplus H^0(\wedge^2 TX)$ we have $(\alpha,\beta,\gamma)^* = (-\alpha,\beta,-\gamma)$.

The *diagonal embedding* of $HT^2(X)$ in $HT^2(X \times X)$ is given by $\alpha \mapsto \pi_1^*(\alpha) + \pi_2^*(\alpha)$. We let the involution $id \otimes (\bullet)^*$ act on $HT^*(X \times X)$ via the Künneth decomposition of the latter.

Given an equivalence $F : D^b(X) \to D^b(Y)$ of the derived categories of two smooth projective varieties X and Y we get the graded ring isomorphism $F^{HT} : HT^*(X) \to HT^*(Y)$ (see [Ca2, Cor. 8.3] and [CBR, Theorem 1.4]). The summand $HT^2(X)$ parametrizes first order deformations of $D^b(X)$ associated to first order deformations of the abelian category of coherent sheaves on X [T]. The summand $HT^1(X)$ is the Lie algebra of the identity component of $\mathrm{Aut}(D^b(X))$, and F^{HT} restricts to the differential of the isomorphism induced by conjugation by F. If $F = f_*$, for an isomorphism $f : X \to Y$, then f_*^{HT} restricts to the summands $H^1(X,\mathcal{O}_X)$ and $H^0(X,TX)$ of $HT^1(X)$ as the homomorphism induced by the direct image functor composed with the isomorphism induced by the natural sheaf isomorphisms $f_*\mathcal{O}_X \to \mathcal{O}_Y$ and $df : f_*TX \to TY$. When X is an abelian variety, the equivalence $\Phi_{\mathcal{P}} : D^b(\hat{X}) \to D^b(X)$ with Fourier-Mukai kernel the Poincaré line bundle $\mathcal{P}$ conjugates autoequivalences associated to translation automorphisms to autoequivalences associated with tensorization by line bundles in Pic^0. Hence, $\Phi_{\mathcal{P}}^{HT} : HT^1(\hat{X}) \to HT^1(X)$ maps the Lie subalgebra $H^0(T\hat{X})$ of the subgroup $\hat{X}$ of translations of $\hat{X}$ to the Lie subalgebra $H^1(\mathcal{O}_X)$ of the subgroup $\mathrm{Pic}^0(X)$ and it maps $H^1(\mathcal{O}_{\hat{X}})$ to $H^0(TX)$.

Lemma 8.4.1. *The composition $\Phi^{HT}\circ(id\otimes(\bullet)^*) : HT^2(X\times X)\rightarrow HT^2(X\times\hat{X})$ maps the diagonal embedding of $HT^2(X)$ into $H^1(T[X\times\hat{X}])\oplus H^2(\mathcal{O}_{X\times\hat{X}})$. The image is the direct sum of the graphs of the following three homomorphisms:*

1. *The graph in $\pi_1^*H^1(TX)\oplus\pi_2^*H^1(T\hat{X})$ of the isomorphism $\Psi^{HT}_{\mathcal{P}^{-1}[n]} : H^1(TX)\rightarrow H^1(T\hat{X})$.*
2. *The graph in $\pi_1^*H^2(\mathcal{O}_X)\oplus H^1(T[X\times\hat{X}])$ of the homomorphism*
$$\begin{array}{rcl} H^2(\mathcal{O}_X) & \rightarrow & H^1(T[X\times\hat{X}]) \\ \eta_1\wedge\eta_2 & \mapsto & \pi_1^*(\eta_1)\wedge\pi_2^*(\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta_2))-\pi_1^*(\eta_2)\wedge\pi_2^*(\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta_1)), \end{array}$$
where $\eta_i\in H^1(\mathcal{O}_X)$, $i=1,2$.
3. *The graph in $\pi_2^*H^2(\mathcal{O}_{\hat{X}})\oplus H^1(T[X\times\hat{X}])$ of the homomorphism*
$$\begin{array}{rcl} H^2(\mathcal{O}_{\hat{X}}) & \rightarrow & H^1(T[X\times\hat{X}]) \\ \eta_1\wedge\eta_2 & \mapsto & -\pi_1^*(\Phi^{HT}_{\mathcal{P}}(\eta_1))\wedge\pi_2^*(\eta_2)+\pi_1^*(\Phi^{HT}_{\mathcal{P}}(\eta_2))\wedge\pi_2^*(\eta_1), \end{array}$$
where $\eta_i\in H^1(\mathcal{O}_{\hat{X}})$, $i=1,2$.

In particular, the image of $HT^2(X)$ in $HT^2(X\times\hat{X})$ projects injectively into the direct summand $H^1(T[X\times\hat{X}])$.

Proof. $\Phi^{HT} : HT^*(X\times X)\rightarrow HT^*(X\times\hat{X})$ is a graded ring isomorphism. $HT^*(X\times X)$ is generated by $HT^1(X\times X)$. The isomorphism $\Phi^{HT}_{\mathcal{P}} : D^b(\hat{X})\rightarrow D^b(X)$, associated to the Poincaré line bundle, maps $H^1(\mathcal{O}_{\hat{X}})$ to $H^0(TX)$ and $H^0(T\hat{X})$ to $H^1(\mathcal{O}_X)$. Hence, its inverse $\Psi^{HT}_{\mathcal{P}^{-1}[n]}$ maps $H^0(TX)$ to $H^1(\mathcal{O}_{\hat{X}})$ and $H^1(\mathcal{O}_X)$ to $H^0(T\hat{X})$.

The isomorphism $(\mu^{-1})_*=\mu^* : D^b(X\times X)\rightarrow D^b(X\times\hat{X})$ induces the isomorphism
$$(\mu^{-1})^{HT}_* : HT^1(X\times X)\rightarrow HT^1(X\times X),$$
where $\mu^{-1}(x,y)=(x-y,y)$. Given $\xi_2,\xi_2\in H^0(TX)$, we have
$$\begin{array}{rcl} (\mu_*^{-1})^{HT}(\pi_1^*(\xi_1\wedge\xi_2)) & = & \pi_1^*(\xi_1\wedge\xi_2). \\ (\mu_*^{-1})^{HT}(\pi_2^*(\xi_1\wedge\xi_2)) & = & [-\pi_1^*\xi_1+\pi_2^*\xi_1]\wedge[-\pi_1^*\xi_2+\pi_2^*\xi_2] \\ & = & \pi_1^*(\xi_1\wedge\xi_2)-\pi_1^*\xi_1\wedge\pi_2^*\xi_2+\pi_1^*\xi_2\wedge\pi_2^*\xi_1+\pi_2^*(\xi_1\wedge\xi_2). \end{array}$$
On the other hand, given $\eta\in H^1(X,\mathcal{O}_X)$,
$$\begin{array}{rcl} \mu^*(\pi_1^*(\eta)) & = & (\pi_1\circ\mu)^*(\eta)=\pi_1^*\eta+\pi_2^*\eta, \\ \mu^*(\pi_2^*(\eta)) & = & \pi_2^*\eta. \end{array}$$
Hence, given $\xi\in H^0(X,TX)$ and $\eta_1,\eta_2\in H^1(X,\mathcal{O}_X)$,
$$\begin{array}{rcl} \mu^*(\pi_1^*(\eta_1\wedge\eta_2)) & = & \pi_1^*(\eta_1\wedge\eta_2)+(\pi_1^*\eta_1\wedge\pi_2^*\eta_2)-(\pi_1^*\eta_2\wedge\pi_2^*\eta_1)+\pi_2^*(\eta_1\wedge\eta_2), \\ \mu^*(\pi_1^*(\xi\wedge\eta)) & = & \pi_1^*\xi\wedge(\pi_1^*\eta+\pi_2^*\eta). \\ \mu^*(\pi_2^*(\xi\wedge\eta)) & = & [-\pi_1^*\xi+\pi_2^*\xi]\wedge\pi_2^*\eta=-(\pi_1^*\xi\wedge\pi_2^*\eta)+\pi_2^*(\xi\wedge\eta). \end{array}$$
Given $\xi_i\in H^0(TX)$, set $\eta_i:=\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\xi_i)\in H^1(\mathcal{O}_{\hat{X}})$, $i=1,2$. $(\mu_*^{-1})^{HT}(\pi_2^*(\xi_1\wedge\xi_2))$ is sent via $(1\boxtimes\Psi_{\mathcal{P}^{-1}[n]})^{HT}$ to
$$\Phi^{HT}(\pi_2^*(\xi_1\wedge\xi_2))=\pi_1^*(\xi_1\wedge\xi_2)-\pi_1^*\xi_1\wedge\pi_2^*\eta_2+\pi_1^*\xi_2\wedge\pi_2^*\eta_1+\pi_2^*(\eta_1\wedge\eta_2).$$

On the other hand, $\Phi^{HT}(\pi_1^*(\xi_1\wedge\xi_2))=\pi_1^*(\xi_1\wedge\xi_2)$. Hence,

(8.4.2) $\Phi^{HT}(-\pi_1^*(\xi_1\wedge\xi_2)+\pi_2^*(\xi_1\wedge\xi_2))=-\pi_1^*\xi_1\wedge\pi_2^*\eta_2+\pi_1^*\xi_2\wedge\pi_2^*\eta_1+\pi_2^*(\eta_1\wedge\eta_2).$

The non-commutative first order deformation $-\pi_1^*(\xi_1\wedge\xi_2)+\pi_2^*(\xi_1\wedge\xi_2)$ of $X\times X$ in $H^0(\wedge^2T(X\times X))$ is mapped to a commutative-gerby deformation of $X\times\hat{X}$.

Let $\eta'_i\in H^1(\mathcal{O}_X)$ and $\xi'_i\in H^0(T\hat{X})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta'_i)=\xi'_i$, $i=1,2$. Then

$$\begin{aligned}\Phi^{HT}(\pi_1^*(\eta'_1\wedge\eta'_2)) &= \pi_1^*(\eta'_1\wedge\eta'_2)+(\pi_1^*\eta'_1\wedge\pi_2^*\xi'_2)-(\pi_1^*\eta'_2\wedge\pi_2^*\xi'_1)+\pi_2^*(\xi'_1\wedge\xi'_2).\\ \Phi^{HT}(\pi_2^*(\eta'_1\wedge\eta'_2)) &= \pi_2^*(\xi'_1\wedge\xi'_2).\end{aligned}$$

Hence,

(8.4.3) $\Phi^{HT}(\pi_1^*(\eta'_1\wedge\eta'_2)-\pi_2^*(\eta'_1\wedge\eta'_2))=\pi_1^*(\eta'_1\wedge\eta'_2)+(\pi_1^*\eta'_1\wedge\pi_2^*\xi'_2)-(\pi_1^*\eta'_2\wedge\pi_2^*\xi'_1).$

Let $\eta'\in H^1(\mathcal{O}_X)$ and $\xi'\in H^0(T\hat{X})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta')=\xi'$. Let $\xi\in H^0(TX)$ and $\eta\in H^1(\mathcal{O}_{\hat{X}})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\xi)=\eta$. Then

$$\begin{aligned}\Phi^{HT}(\pi_1^*(\xi\wedge\eta')) &= \pi_1^*(\xi\wedge\eta')+(\pi_1^*\xi\wedge\pi_2^*\xi')).\\ \Phi^{HT}(\pi_2^*(\xi\wedge\eta')) &= -(\pi_1^*\xi\wedge\pi_2^*\xi')+\pi_2^*(\eta\wedge\xi').\\ \Phi^{HT}(\pi_1^*(\xi\wedge\eta'+\pi_2^*(\xi\wedge\eta')) &= \pi_1^*(\xi\wedge\eta')+\pi_2^*(\eta\wedge\xi').\end{aligned}$$

So Φ^{HT} maps the diagonal commutative first order deformations of $X\times X$ to commutative first order deformations of $X\times\hat{X}$. Furthermore, it maps the anti-diagonal non-commutative and gerby deformations of $X\times X$ to commutative and gerby deformations of $X\times\hat{X}$, by Equations (8.4.2) and (8.4.3). □

Let F_1, F_2, and E be as in Equation (8.3.4). Note that $ch(F_1)=ch(F_2)$. Hence, $\ker(ob_{F_1})=\ker(ob_{F_2})$, by Corollary 8.2.12.

Corollary 8.4.2. *The isomorphism* $\Phi^{HT}\circ(id\otimes(\bullet)^*):HT^2(X\times X)\to HT^2(X\times\hat{X})$ *maps the diagonal embedding of the* 9*-dimensional subspace* $\ker(ob_{F_1})$ *of* $HT^2(X)$ *to a* 9*-dimensional subspace of* $H^1(T[X\times\hat{X}])\oplus H^2(\mathcal{O}_{X\times\hat{X}})$ *of commutative and gerby deformations in the kernel of* ob_E*.*

Proof. Step 1: We claim that $\ker(ob_{F_i})=\ker(ob_{F_i^\vee})$. The kernel of of_{F_i} is the annihilator of $ch(F_i)$, which is the kernel of the following homomorphism.

$$\begin{array}{c}H^2(\mathcal{O}_X)\\ \oplus\\ H^1(TX)\\ \oplus\\ H^0(\wedge^2TX)\end{array}\ \xrightarrow{\left(\begin{array}{ccc}1 & \Theta & -(d/2)\Theta^2\\ \Theta & -(d/2)\Theta^2 & -(d/6)\Theta^3\end{array}\right)}\ \begin{array}{c}H^2(\mathcal{O}_X)\\ \oplus\\ H^3(\Omega^1_X)\end{array}$$

The subspace annihilating $ch(F_i^\vee)$ is the kernel of the homomorphism obtained by replacing the above matrix by

$$\left(\begin{array}{cc}-1 & 0\\ 0 & 1\end{array}\right)\left(\begin{array}{ccc}1 & \Theta & -(d/2)\Theta^2\\ \Theta & -(d/2)\Theta^2 & -(d/6)\Theta^3\end{array}\right)\left(\begin{array}{ccc}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{array}\right)$$

Hence, it suffices to show that the kernel is the direct sum of the subspace of $H^1(TX)$ annihilating Θ and the subspace of $H^2(\mathcal{O}_X)\oplus H^0(\wedge^2 TX)$ in the kernel of $(1,-\frac{d}{2}\Theta^2)$. Indeed, both are 9 dimensional, and so it suffices to prove the inclusion of this direct sum in the subspace annihilating $ch(F_i)$. This follows from the fact that (i) the subspace of $H^1(TX)$ annihilating Θ is equal to the subspace annihilating Θ^2, and (ii) the subspace of $H^2(\mathcal{O}_X)\oplus H^0(\wedge^2 TX)$ in the kernel of $(1,-\frac{d}{2}\Theta^2)$ is equal to the subspace in the kernel of $(\Theta,-\frac{d}{6}\Theta^3)$. Fact (i) is easy to verify. Fact (ii) follows from the identity $\xi\rfloor c\Theta^n = (-1)^i nc(\xi\rfloor\Theta)\Theta^{n-1}$, for $\xi\in H^0(TX)$ and $c\in H^i(\mathcal{O}_X)$, $i\geq 0$, and the observation that $c(\xi\rfloor\Theta)$ belongs to $H^{i+1}(\mathcal{O}_X)$. Indeed, the identity implies that the homomorphism $(\Theta,-\frac{d}{6}\Theta^3)$ is the composition

$$\begin{array}{c} H^2(\mathcal{O}_X)\\ \oplus \\ H^0(\wedge^2 TX)\end{array} \xrightarrow{\left(\begin{array}{cc} 1 & -(d/2)\Theta^2\end{array}\right)} H^2(\mathcal{O}_X) \xrightarrow{\Theta\cup} H^3(\Omega^1_X)$$

and cup product with Θ is an injective homomorphism.

Step 2: If e_α annihilates $ch(F_1)$, then it annihilates $ch(F_2)$ and also $ch(F_2^\vee)$, by Step 1, and so e_{α^*} annihilates $ch(F_2)$. Hence, $e_{\pi_1^*(\alpha)+\pi_2^*(\alpha^*)}$ annihilates $ch(F_1\boxtimes F_2)$. It follows that $e_{\Phi^{HT}(\pi_1^*(\alpha)+\pi_2^*(\alpha^*))}$ annihilates $ch(E)$, by [CBR, Theorem 1.4]. The statement thus follows from Lemmas 8.4.1 and 8.3.1(2) $\square$

Remark 8.4.3. The image of the diagonal embedding in $HT^2(X\times X)$, of the kernel in $HT^2(X)$ of ob_{F_1}, is mapped via $\Phi^{HT}\circ(id\otimes(\bullet)^*)$ into a subspace of $H^2(\mathcal{O}_{X\times\hat{X}})\oplus H^1(T(X\times\hat{X}))$, which projects onto the tangent space in $H^1(T(X\times\hat{X}))$ of the moduli space of abelian varieties of Weil type. This will follow from Lemma 9.3.9, which shows that contraction with $\exp(-c_1(E)/\mathrm{rank}(E))$ induces an automorphism of $HT^2(X\times\hat{X})$ mapping the kernel of ob_E into the subspace of $H^1(T(X\times\hat{X}))$ tangent to the moduli space of abelian varieties of Weil type. The above automorphism is unipotent, leaving each of $H^2(\mathcal{O}_{X\times\hat{X}})\oplus H^1(T(X\times\hat{X}))$ and $H^2(\mathcal{O}_{X\times\hat{X}})$ invariant and inducing the identity on the graded summand. Hence, the automorphism restricts to the kernel of ob_E as the projection to $H^1(T(X\times\hat{X}))$.

Remark 8.4.4. All the results of Section 8 remain valid after interchanging the sheaves $F_1 := \mathcal{I}_{\cup_{i=1}^{d+1}C_i}(\Theta)$ and $F_2 := \mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}(\Theta)$. The two are interchanged by the involution $\iota: X\to X$ sending $L\in X=\mathrm{Pic}^2(C)$ to $\omega_C\otimes L^{-1}$. The cohomological action of ι corresponds to the element in the center of $\mathrm{Spin}(V)$ acting as the isentity on S^+ and by multiplication by -1 on V and S^-. It acts on $HT^{ev}(X)$, and so on $HT^2(X)$, as the identity and on $HT^{odd}(X)$ by multiplication by -1.

Remark 8.4.5. Note that when $n=2$ and F is a simple sheaf on an abelian surface X, then the kernel of ob_F is 5-dimensional and Orlov's derived equivalence maps it to a subspace of $H^2(\mathcal{O}_{X\times\hat{X}})\oplus H^1(T(X\times\hat{X}))$, which projects onto the tangent space in $H^1(T(X\times\hat{X}))$ of the moduli space of intermediate jacobians associated to the 3-rd cohomology of generalized kummers [M2].

9. A reflexive sheaf over $X \times \hat{X}$ with $\mathrm{Spin}(V)_P$-invariant characteristic classes

Let C be a non-hyperelliptic curve of genus 3. Let $C_{\ell_i} = AJ(C) + \ell_i$, $1 \leq i \leq d+1$, be the translate in $X = \mathrm{Pic}^2(C)$ of the Abel-Jacobi image $AJ(C) \subset \mathrm{Pic}^1(C)$ of C by $\ell_i \in \mathrm{Pic}^1(C)$. In Section 9.1 we show that $H^0(X, \mathcal{I}_{\cup_{i=1}^{d+1} C_i}(2\Theta) \otimes L)$ vanishes for all $L \in \mathrm{Pic}^0(X)$, provided the intersection $\bigcap_{1\leq i<j\leq d+1} \tau_{-\ell_i - \ell_j}(\Theta)$ is empty. Furthermore, the latter condition holds for generic C, if the set $\{\ell_i\}_{i=1}^{d+1}$ is an orbit under translations by elements of a cyclic subgroup of $\mathrm{Pic}^0(C)$ of order $d+1$. Set $F_1 := \mathcal{I}_{\cup_{i=1}^{d+1} C_i}(\Theta)$ and $F_2 := \mathcal{I}_{\cup_{i=1}^{d+1} \Sigma_i}(\Theta)$. Set $\mathcal{G} := \Phi(F_2 \boxtimes F_1)[-3]$, where Φ is Orlov's derived equivalence. In Section 9.2 we show that $\mathcal{E} := \mathcal{G}^\vee[-1]$ is a reflexive sheaf over $X \times \hat{X}$ of rank $8d$, for a generic choice of the curves C_i and Σ_j. Furthermore, $\mathcal{E}$ is locally free away from $(d+1)^2$ smooth surfaces in $X \times \hat{X}$. In Section 9.3 we choose the set $\{C_i\}_{i=1}^{d+1}$ to consist of translates of C_1 by a cyclic subgroup G_1 of $\mathrm{Pic}^0(C)$ and choose the set $\{\Sigma_i\}_{i=1}^{d+1}$ similarly, for a cyclic subgroup G_2 of $\mathrm{Pic}^0(C)$. We show that a tensor product of the reflexive sheaf $\mathcal{E}$ with a suitable twisted line bundle descends to a semiregular reflexive twisted sheaf $\mathcal{B}$ over a quotient Y of $X \times \hat{X}$ by a group $\bar{G}$ of translations, where $\bar{G} \cong G_1 \times G_2$. We then prove Theorem 1.5.1, stating the algebraicity of the Hodge-Weil classes over abelian sixfolds of Weil type of discriminant -1, using the version of the Semiregularity Theorem for twisted sheaves proved in Section 7.4.2.

9.1. **A general position assumption.** Keep the notation of Section 8. Let $\Sigma_t \subset X$ be the image of C_t under the involution ι of $X = \mathrm{Pic}^2(C)$ sending a line bundle L to $\omega_C \otimes L^{-1}$. Then Σ_t and C_t are not algebraically equivalent for a generic non-hyperelliptic C. Furthermore, the cycle $C_t - \Sigma_t$ is non-torsion in the group of algebraic cycles modulo algebraic equivalence, by [Ce]. We will not need these facts and assume only that C is non-hyperelliptic.

Choose a point $p \in C$ and use it to identify X and $\mathrm{Pic}^0(C)$ via tensorization with $\mathcal{O}_C(2p)$ endowing X with a group structure. Set

$$C_p := AJ(C) + p = \{\mathcal{O}_C(p+q) \ : \ q \in C\}.$$

Set $C_i = \tau_{s_i}(C_p)$, for a point $s_i \in X$, $1 \leq i \leq d+1$. Set $\Sigma_p := \iota(C_p)$ and $\Sigma_i = \tau_{t_i}(\Sigma_p)$, for a point $t_i \in X$, $1 \leq i \leq d+1$. Assume that the curves C_i are pairwise disjoint and so are the Σ_i.

Assumption 9.1.1. *Assume that $H^0(X, \mathcal{I}_{\cup_{i=1}^{d+1} C_i}(2\Theta) \otimes L)$ vanishes, for all $L \in \mathrm{Pic}^0(X)$.*

Given a line bundle L of degree k, set $C_L := \{L(p) \ : \ p \in C\} \subset \mathrm{Pic}^{k+1}(C)$.

Lemma 9.1.2. *Assumption 9.1.1 holds for $d \geq 3$ for a generic choice of C_i's. Given a subset $\{\ell_i\}_{i=1}^{d+1}$ of $\mathrm{Pic}^1(C)$ the assumption holds for $\{C_{\ell_i}\}_{i=1}^{d+1}$, provided*

$$\bigcap_{1\leq i<j\leq d+1} \tau_{-\ell_i-\ell_j}(\Theta) = \emptyset. \tag{9.1.1}$$

Proof. Step 1: Note first that for a generic choice of ℓ_i's the intersection (9.1.1) is empty. It suffices to prove it for $d = 3$. Now the intersection of generic 4 translates of Θ is empty. So it suffices to observe that the morphism $t : \mathrm{Pic}^1(C) \to \mathrm{Pic}^2(C)$ given by

$$t(\ell_1, \ell_2, \ell_3, \ell_4) = (\ell_1 + \ell_2, \ell_1 + \ell_3, \ell_1 + \ell_4, \ell_2 + \ell_3)$$

is surjective. Indeed, set $T = (t_1, t_2, t_3, t_4)$ and define $f : \mathrm{Pic}^2(C)^4 \to \mathrm{Pic}^2(C)$ by $f(T) = t_1 + t_2 - t_4$ and $u : \mathrm{Pic}^2(C)^4 \to \mathrm{Pic}^2(C)^4$ by

$$u(T) = (f(T), 2t_1 - f(T), 2t_2 - f(T), 2t_3 - f(T)).$$

Then $f(t(\ell_1, \ell_2, \ell_3, \ell_4)) = 2\ell_1$ and $u(t(\ell_1, \ell_2, \ell_3, \ell_4)) = (2\ell_1, 2\ell_2, 2\ell_3, 2\ell_4)$. It follows that $u \circ t$ is surjective, and hence so is t, as the image of t must be 4 dimensional.

Step 2: We show next that a divisor D in $|2\Theta|$ contains the translate C_L, $L \in \mathrm{Pic}^1(C)$, if and only if D is the divisor D_E associated to some semistable rank 2 vector bundle E on C with trivial determinant and L^{-1} is a subsheaf of E. Recall that for a semistable vector bundle E on C with trivial determinant the set

$$\{M \in \mathrm{Pic}^2(C) \ : \ H^0(C, E \otimes M) \neq 0\}$$

is the set theoretic support of a divisor D_E in $|2\Theta|$, which depends only on the S-equivalence class of E (see [NR]). The inclusion $C_L \subset D_E$ is clear, if L^{-1} is a subsheaf of E. Conversely, assume that C_L is contained in D, $D \in |2\Theta|$. Then D belongs to the projective subspace $\mathbb{P}[H^0(\mathcal{I}_{C_L}(2\Theta))]$ of $|2\Theta|$. Now $\mathbb{P}[H^0(\mathcal{I}_{C_L}(2\Theta))]$ is 3-dimensional (we postpone the proof to Lemma 9.2.8). So it suffices to prove the claim that the subset of the moduli space of S-equivalence classes of semistable vector bundles of rank 2 and trivial determinant, which has a representative containing L^{-1} as a subsheaf, is isomorphic to $\mathbb{P}^3$. Indeed, every such equivalence class $[E]$ is represented by an extension class in $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}) \cong \mathbb{P}H^0(L^2 \otimes \omega_C)^* \cong \mathbb{P}^3$. This is clear if $[E]$ is represented by a vector bundle containing L^{-1} as a saturated subsheaf. The semi-stable but unstable S-equivalence classes in $\mathbb{P}\mathrm{Ext}^1(L, L^{-1})$ are represented by the set $L^{-1}(q) \oplus L(-q)$, $q \in C$, by [B, Lemma 3.6(b)]. This is precisely the set of S-equivalence classes of rank 2 semistable vector bundles of trivial determinant containing L^{-1} as an unsaturated subsheaf. Finally, the map $E \mapsto D_E$ is a linear embedding of $\mathbb{P}\mathrm{Ext}^1(L, L^{-1})$ in $|2\Theta|$, by [B, Lemma 3.7], proving the claim.

Step 3: We observe next that if E is a semistable vector bundle with trivial determinant on C, L_1 and L_2 are two line bundles of degree 1 admitting embeddings $\iota_k : L_k^{-1} \to E$ as subsheaves, and the curves C_{L_1} and C_{L_2} are disjoint in $\mathrm{Pic}^2(C)$, then the line bundle $L_1 \otimes L_2$ is effective. We have two cases. Case 1: If L_1^{-1} is a subbundle of E we get the short exact sequence

$$0 \to L_1^{-1} \xrightarrow{\iota_1} E \xrightarrow{j_1} L_1 \to 0$$

and $j_1 \circ \iota_2 : L_2^{-1} \to L_1$ does not vanish, hence L_1 is isomorphic to $L_2^{-1}(p+q)$, for some points $p, q \in C$.

Case 2: If L_1^{-1} is only a subsheaf, then $L_1^{-1}(p)$ is a subbundle, for some point $p \in C$. We get the short exact sequence

$$0 \to L_1^{-1}(p) \to E \xrightarrow{j_1} L_1(-p) \to 0.$$

If L_2^{-1} is a subsheaf of $L_1^{-1}(p)$, then L_2 is isomorphic to $L_1(q-p)$, for some point $q \in C$, and the curves C_{L_1} and C_{L_2} both contain $L_1(q) \cong L_2(p)$ and so are not disjoint. Hence, $j_1 \circ \iota_2$ does not vanish and L_2^{-1} is isomorphic to $L_1(-p-q)$, for some point $q \in C$.

Step 4: Assume that the intersection (9.1.1) is empty. Then there does not exist a point $t \in \mathrm{Pic}^0(C)$, such that $2t + \ell_i + \ell_j \in \Theta$, for all $1 \leq i < j \leq d+1$. Thus there does not exist a semi-stable vector bundle E of trivial determinant, such that D_E contains the union $\bigcup_{i=1}^{d+1} C_{\ell_i + t}$, for some $t \in \mathrm{Pic}^0(C)$, by Step 3. Hence, $H^0(X, \mathcal{I}_{\cup_{i=1}^{d+1} C_{\ell_i+t}}(2\Theta))$ vanishes, for all $t \in \mathrm{Pic}^0(C)$, by Step 2. It follows that $H^0(X, \mathcal{I}_{\cup_{i=1}^{d+1} C_{\ell_i}}(\tau_t^*(2\Theta)))$ vanishes, for all $t \in \mathrm{Pic}^0(C)$. Finally, the morphism $\mathrm{Pic}^0(C) \to \mathrm{Pic}(X)$, given by $t \mapsto \tau_t^*(2\Theta)$, is an isogeny onto the connected component of 2Θ. □

Remark 9.1.3. Consider the rank 4 vector bundle $\mathcal{U}$ over $\mathrm{Pic}^1(C)$ with fiber $\mathcal{U}_L := H^0(X, \mathcal{I}_{C_L}(2\Theta))$ over $L \in \mathrm{Pic}^1(C)$. We have seen in the proof above that $\mathbb{P}(\mathcal{U}_L)$ is naturally isomorphic to $\mathbb{P}H^0(C, L^2 \otimes \omega_C)$. Hence, one gets a morphism $\epsilon : \mathbb{P}(\mathcal{U}) \to |2\Theta|$, by [B, Lemma 3.7]. The morphism is generically finite of degree 8 onto its image, which is the Coble quartic and is isomorphic to the moduli space of equivalence classes of semistable rank 2 vector bundles of trivial determinant over C, by [NR] and [Pa, Sec. 4.1]. The Zariski closed subset $\epsilon^{-1}(\epsilon(\mathbb{P}(\mathcal{U}_L)))$ is thus reducible of dimension ≥ 3. Let $\pi : \mathbb{P}(\mathcal{U}) \to \mathrm{Pic}^1(C)$ be the natural projection. The proof above shows that the components of $\epsilon^{-1}(\epsilon(\mathbb{P}(\mathcal{U}_L)))$ other that $\mathbb{P}(\mathcal{U}_L)$ all lie in $\pi^{-1}(\tau_{L^{-1}}(\Theta))$ and so do not surject onto $\mathrm{Pic}^1(C)$. Pauly shows that the generic rank 2 stable vector bundle E of trivial determinant contains the inverses of 8 distinct line bundles L_i, $1 \leq i \leq 8$, in $\mathrm{Pic}^1(C)$ and $\otimes_{i=1}^8 L_i \cong \omega_C^2$ [Pa, Lemma 4.2]. The divisor D_E, for such E, contains precisely 8 translates C_{L_i}, $1 \leq i \leq 8$, of C.

Lemma 9.1.4. *Let G be a cyclic subgroup of $\mathrm{Pic}^0(C)$ of order $d+1$, $d \geq 3$. The emptiness condition (9.1.1) holds and the curves C_{L_i} are pairwise disjoint for every G-orbit $\{L_i\}_{i=1}^{d+1}$ in $\mathrm{Pic}^1(C)$, for a generic C.*

Proof. The emptiness condition (9.1.1) would follow from the emptiness of $\bigcap_{g \in G} \tau_g(\Theta)$. The latter emptiness condition is open in the moduli space of pairs (C, G) and so it suffices to verify it for a boundary pair, where C is a chain $E_1 \cup E_2 \cup E_3$ of three elliptic curves E_i, $1 \leq i \leq 3$. Let G_i be a cyclic group of order $d+1$ of $\mathrm{Pic}^0(E_i)$, and let G be a diagonal embedding of $\mathbb{Z}/(d+1)\mathbb{Z}$ in $G_1 \times G_2 \times G_3 \subset \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) \times \mathrm{Pic}^0(E_3) \cong \mathrm{Pic}^0(C)$. Assume that E_2 intersects E_1 at the point p_0 of E_2 and E_2 intersect E_3 and the point p_1 of E_2, and $p_1 - p_0$ does not belong to G_2. We will explain below that in this case $\Theta = \cup_{i=1}^3 D_i$, where

$$
\begin{aligned}
D_1 &:= \{\mathcal{O}_{E_1}\} \times \mathrm{Pic}^2(E_2) \times \mathrm{Pic}^0(E_3), \\
D_2 &:= \mathrm{Pic}^0(E_1) \times \{\mathcal{O}_{E_2}(p_0 + p_1)\} \times \mathrm{Pic}^0(E_3), \\
D_3 &:= \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^2(E_2) \times \{\mathcal{O}_{E_3}\}.
\end{aligned}
\tag{9.1.2}
$$

Choose a generator g of G.

$$
\bigcap_{k=1}^{d+1} \tau_{kg}(\Theta) = \bigcup_{(i_0, \ldots, i_d) \in \{1,2,3\}^{d+1}} \bigcap_{k=1}^{d+1} \tau_{kg}(D_{i_k}).
$$

Each intersection $\bigcap_{k=1}^{d+1}\tau_{kg}(D_{i_k})$ is empty, as $i_j=i_k$ for some $j<k$, since $d+1>3$, and $D_i\cap\tau_{kg}(D_i)=\emptyset$, for $0<k<d+1$, for all i.

We show next that the G-orbit of the Abel-Jacobi image of C consists of pairwise distinct curves. Consider the following embedded copy C' of C in $\mathrm{Pic}^{(0,1,0)}(C)$.

$$\begin{array}{lcl}\left[\mathrm{Pic}^0(E_1)\times\{\mathcal{O}_{E_2}(p_0)\}\times\{\mathcal{O}_{E_3}\}\right] & \cup & \left[\{\mathcal{O}_{E_1}\}\times\mathrm{Pic}^1(E_2)\times\{\mathcal{O}_{E_3}\}\right]\\ & \cup & \left[\{\mathcal{O}_{E_1}\}\times\mathcal{O}_{E_2}(p_1)\}\times\mathrm{Pic}^0(E_3)\right].\end{array}$$

Denote the i-th component above by C'_i. Then $C'_1\cap C'_2=(\mathcal{O}_{E_1},\mathcal{O}_{E_2}(p_0),\mathcal{O}_{E_3})$, $C'_2\cap C'_3=(\mathcal{O}_{E_1},\mathcal{O}_{E_2}(p_1),\mathcal{O}_{E_3})$, and C'_1 and C'_3 are disjoint. The embedding $AJ:C\to C'\subset\mathrm{Pic}^{(0,1,0)}(C)$ is given by

$$p\mapsto\left\{\begin{array}{cll}(\mathcal{O}_{E_1}(p-p_0),\mathcal{O}_{E_2}(p_0),\mathcal{O}_{E_3}) & \text{if} & p\in E_1,\\ (\mathcal{O}_{E_1},\mathcal{O}_{E_2}(p),\mathcal{O}_{E_3}) & \text{if} & p\in E_2,\\ (\mathcal{O}_{E_1},\mathcal{O}_{E_2}(p_1),\mathcal{O}_{E_3}(p-p_1)) & \text{if} & p\in E_3.\end{array}\right. \tag{9.1.3}$$

For $0<k<d+1$, $C'_i\cap\tau_{kg}(C'_i)=\emptyset$, for all i, and $C'_i\cap\tau_{kg}(C'_j)=\emptyset$, for $\{i,j\}=\{1,2\}$ and $\{2,3\}$. The intersection is empty also for $\{i,j\}=\{1,3\}$, since $p_1-p_0\notin G_2$. Hence, the translates $\tau_{kg}(C')$, $0\leq k\leq d$, are pairwise disjoint.

We prove next that C' is the limit of Abel-Jacobi images of smooth genus 3 curves in a flat family that degenerates to C. Observe that the curve C' is precisely the Brill-Noether locus of $L\in\mathrm{Pic}^{(0,1,0)}(C)$ with $h^0(L)\neq 0$. Indeed, if $L=(L_1,L_2,L_3)$ and s is a non-zero global section of L, then s is not identically zero on E_2. The line bundle L_2 is isomorphic to $\mathcal{O}_{E_2}(p)$, for a unique point $p\in E_2$. If $p\notin\{p_0,p_1\}$, then s does not vanish at p_0 and p_1 and so L_1 and L_2 must both be trivial, so that L is in C'_2. If $p=p_0$, then s vanishes at p_0 and so it must be identically zero on E_1, but non-zero on E_3. Hence, L_1 is arbitrary, but L_3 is trivial, and so L is in C'_1. Similarly, if $p=p_1$, then L is in C'_3. Choose a family $\pi:\mathcal{C}\to S$ over a smooth one-dimensional analytic base S with special fiber C over $0\in S$ and generic fiber a smooth genus 3 curve. Let $q:S\to\mathcal{C}$ be a section with $q(0)$ a point of $E_2\setminus\{p_0,p_1\}$. The section q determines a section $Q:S\to\mathrm{Pic}(\mathcal{C}/S)$ with value $\mathcal{O}_{C_s}(q(s))$ over $s\in S$, hence a family of abelian varieties $\Pi:\mathcal{J}\to S$ with connected fibers, whose generic fiber is $\mathrm{Pic}^1(C_s)$ and its special fiber is $\mathrm{Pic}^{(0,1,0)}(C)$. Over $\mathcal{C}\times_S\mathrm{Pic}^0(\mathcal{C}/S)$ we have the relative (normalized) Poincaré line bundle $\mathcal{P}_0$. Let $\tau_Q:\mathrm{Pic}^0(\mathcal{C}/S)\to\mathcal{J}$ be the isomorphism of translation by Q. Let $f_1:\mathcal{C}\times_S\mathcal{J}\to\mathcal{J}$ and $f_2:\mathcal{C}\times_S\mathcal{J}\to\mathcal{C}$ be the projections. Translating $\mathcal{P}_0$ by the section Q and tensoring by the pullback of the line bundle $\mathcal{O}_{\mathcal{C}}(q(S))$ over $\mathcal{C}$ and we get a relative Poincaré line bundle $\mathcal{P}:=(id\times\tau_Q)_*(\mathcal{P}_0)\otimes f_2^*\mathcal{O}_{\mathcal{C}}(q(S))$ over $\mathcal{C}\times_S\mathcal{J}$. The coherent torsion sheaf $R^1f_*(\omega_{f_1}\otimes\mathcal{P}^{-1})$ is supported as a line bundle on a relative curve $\mathcal{C}'$ over S, by Cohomology and Base Change. The generic fiber of $\mathcal{C}'$ is the Abel-Jacobi image of C_s and the special fiber is C'.

Finally observe that Θ is the image of $\mathrm{Sym}^2(C')$ via the addition morphism

$$\mathrm{Sym}^2(\mathrm{Pic}^{(0,1,0)}(C))\to\mathrm{Pic}^{(0,2,0)}(C)$$

(the symmetric square of each irreducible component of C' contracts to a curve in Θ). $\square$

9.2. **A reflexive secant$^{\boxtimes 2}$-sheaf over** $X\times\hat{X}$**.** We continue to assume Assumption 9.1.1. Set

$$F_1 := \mathcal{I}_{\cup_{i=1}^{d+1}C_i}(\Theta) \ \text{ and } \ F_2 := \mathcal{I}_{\cup_{i=1}^{d+1}\Sigma_i}(\Theta). \tag{9.2.1}$$

Note that $ch(F_1)=ch(F_2)$. Let $\mathcal{F}_2 := \pi_{12}^*(a^*(F_2))\otimes\pi_{13}^*\mathcal{P}^{-1}$ be the sheaf over $X\times X\times\hat{X}$ using the notation of (6.2.1).

Assumption 9.2.1. (1) *We choose the curves C_i and Σ_j, so that the intersection of any four of the surfaces $\Theta_{i,j} := \Sigma_j - C_i$ in X is empty and the triple intersections are zero dimensional.*
(2) *The $(d+1)^2$ points t_j+s_i, $1\leq i,j\leq d+1$, are distinct.*[20]

Let $f_{i,j}: C_i\times\Sigma_j\to X\times\hat{X}$ be the morphism given by $f_{i,j}(x,y)=(y-x, L_{i,j}(x-y))$, where $L_{i,j}(x-y)\in \mathrm{Pic}^0(X)$ is the line bundle

$$L_{i,j}(x-y) := \mathcal{O}_X(\tau_{2(x-y)-t_j-s_i}(\Theta)-\Theta). \tag{9.2.2}$$

Denote by $\tilde{\Theta}_{i,j}$ the image of $f_{i,j}$ and observe that it is isomorphic to $\Theta_{i,j}$. Note that the surface $\pi_X(\tilde{\Theta}_{i,j})=\Theta_{i,j}$ is a translate[21] of Θ and is thus smooth. Assumption 9.2.1(2) implies that the surfaces $\tilde{\Theta}_{i,j}$ are pairwise disjoint. In particular, the union

$$\tilde{\Theta} := \cup_{1\leq i,j\leq d+1}\tilde{\Theta}_{i,j}$$

is smooth.

Proposition 9.2.2. (1) *The sheaf cohomology $R^i\pi_{23,*}(\pi_1^*F_1\otimes\mathcal{F}_2)$ of the object*

$$\mathcal{G} := \Phi(F_2\boxtimes F_1)[-3] \stackrel{(6.2.3)}{=} R\pi_{23,*}(\pi_1^*F_1\otimes\mathcal{F}_2)$$

vanishes, for $k=0$ and for $k>2$. The sheaf $\mathcal{G}_1 := R^1\pi_{23,*}(\pi_1^*F_1\otimes\mathcal{F}_2)$ over $X\times\hat{X}$ is reflexive of rank $8d$ and it is locally free away from $\tilde{\Theta}$ and $\tilde{\Theta}$ is the set theoretic support of the sheaf $\mathcal{G}_2 := R^2\pi_{23,*}(\pi_1^*F_1\otimes\mathcal{F}_2)$.
(2) *The object $\mathcal{G}^\vee[-1]$ is represented by the reflexive coherent sheaf $\mathcal{E}$ of rank $8d$, which is isomorphic to $\mathcal{G}_1^*$ and is hence locally free away from $\tilde{\Theta}$. The sheaf $\mathcal{G}_2$ is isomorphic to $\mathcal{E}xt^1(\mathcal{E},\mathcal{O}_{X\times\hat{X}})$.*

Remark 9.2.3. Note that the object $\mathcal{G}$ is related to the object E in (8.3.4) by interchanging F_1 and F_2. All the results of section 8 for E holds for $\mathcal{G}$ as well, by Remark 8.4.4.

The proof of Proposition 9.2.2 requires the following lemmas.

Lemma 9.2.4. (1) *Given $t\in\mathrm{Pic}^0(X)$, the intersection $C_i\cap\tau_t(\Sigma_j)$ is either empty, or a subscheme of length 2.*
(2) *The intersection subscheme $C_i\cap\tau_t(\Sigma_j)$ has length 2, if and only if there exists $u\in\mathrm{Pic}^0(C)$, such that $C_i\cup\tau_t(\Sigma_j)$ is contained in $\tau_u(\Theta)$.*

[20] Recall that $C_i=\tau_{s_i}(C_p)$ and $\Sigma_j=\tau_{t_j}(\Sigma_p)$.

[21] Let q_1,q_2 be points of C. Consider the special case where τ_{s_i} and τ_{t_j} are both the identity. The morphism $\pi_X\circ f_{i,j}: C_p\times\Sigma_p\to X$ sends the pair of degree 2 line bundles $(\mathcal{O}_C(p+q_1),\omega_C(-p-q_2))$ to $\omega_C(-2p-q_1-q_2)$, hence $\pi_X\circ f_{i,j}$ is a branched double cover onto its image $\tau_{-2p}(\Theta)$.

(3) *If $C_i \cup \tau_t(\Sigma_j)$ is contained in $\tau_u(\Theta)$, then one of the following holds.*
   (a) *The union $C_i \cup \tau_t(\Sigma_j)$ is the complete intersection $\tau_u(\Theta) \cap \tau_{u'}(\Theta)$ of a unique pair of translates of Θ. Furthermore, $(\tau_u \iota \tau_{-u})(C_i) \neq \tau_t(\Sigma_j)$.*
   (b) *The divisor $\tau_u(\Theta)$ is the unique translate of Θ containing $C_i \cup \tau_t(\Sigma_j)$. Furthermore, $(\tau_u \iota \tau_{-u})(C_i) = \tau_t(\Sigma_j)$ and the canonical line bundle of $C_i \cup \tau_t(\Sigma_j)$ is the restriction of $\mathcal{O}_X(2\tau_u(\Theta))$.*

Proof. Given $p \in C$, set $C_p := C_{AJ(p)}$. We may assume, without loss of generality, that $C_i = C_p$, for some $p \in C$. Let s be a point of $\mathrm{Pic}^0(C)$, such that $\tau_t(\Sigma_j) = \tau_s(\Sigma_p)$.

Step 0: We prove part (3) with the exception of the uniqueness in part (3a) which is postponed to Step 4 below. Observe the equivalence

$$C_p \cup \tau_s(\Sigma_p) \subset \tau_u(\Theta) \Leftrightarrow C_p \cup \tau_s(\Sigma_p) \subset \tau_{s-u}(\Theta). \tag{9.2.3}$$

Indeed, the right inclusion is obtained by applying $\tau_s \circ \iota$ to both sides of the left inclusion and using the equalities $\iota(\Theta) = \Theta$ and $\tau_s \circ \iota = \iota \circ \tau_{-s}$.

If $s \neq 2u$, then the union $C_p \cup \tau_s(\Sigma_p)$ is contained in the two distinct translates $\tau_u(\Theta)$ and $\tau_{s-u}(\Theta)$ and so the union must be their complete intersection, as the cohomology classes of both are equal. If $\tau_u(\Theta)$ is the unique translate of Θ containing $C_p \cup \tau_s(\Sigma_p)$, then $s = 2u$ and so $(\tau_u \iota \tau_{-u})(C_p) = \tau_{2u}(\iota(C_p)) = \tau_s(\Sigma_p)$. If there exist two distinct translates of Θ containing $C_p \cup \tau_s(\Sigma_p)$, then the latter is their complete intersection, by the equality of the cohomology classes.

Assume that $(\tau_u \iota \tau_{-u})(C_p) = \tau_s(\Sigma_p)$. Then $\tau_{2u}(\Sigma_p) = \tau_s(\Sigma_p)$. Hence, $s = 2u$. We will see in Step 3 that $\tau_u(\Theta)$ is the unique translate of Θ containing both C_p and $\tau_{2u}(\Sigma_p)$. This would prove the inequality in part 3a.

In both cases $Z' := C_i \cup \tau_t(\Sigma_j)$ is a divisor on a smooth surface $\tau_u(\Theta)$, and so its canonical sheaf is a line bundle. In case (3a) the canonical line bundle $\omega_{Z'}$ is the restriction of $\mathcal{O}_X(\tau_u(\Theta)+\tau_{u'}(\Theta))$, as the normal bundle is the restriction of $\mathcal{O}_X(\tau_u(\Theta)) \oplus \mathcal{O}_X(\tau_{u'}(\Theta))$. Hence, $H^0(Z', \omega_{Z'}(-\tau_u(\Theta) - \tau_{u'}(\Theta)))$ is one dimensional. Case (3b) is a limit case and we conclude that $H^0(Z', \omega_{Z'}(-2\tau_u(\Theta)))$ does not vanish, by semi-continuity. The line bundle $\omega_{Z'}(-2\tau_u(\Theta))$ restricts to each irreducible component of Z' as a line bundle of degree 0, hence the existence of a non-zero global section implies that it is the trivial line-bundle.

Step 1: Let $q \in C$ be a point not equal to p and set $y = p - q$. We prove first the equalities

$$\Theta \cap \tau_y(\Theta) = C_p \cup \Sigma_q, \tag{9.2.4}$$

$$C_p \cap \Sigma_q = \{\mathcal{O}_C(p+r), \mathcal{O}_C(p+t)\}, \tag{9.2.5}$$

such that $\omega_C \cong \mathcal{O}_C(p+q+r+t)$. The points r and t are the two other points on the intersection of the line through $\varphi_{\omega_C}(p)$ and $\varphi_{\omega_C}(q)$ with the canonical curve $\varphi_{\omega_C}(C)$. The curves C_p and Σ_q are tangent at $\mathcal{O}_C(p+r)$, if $r = t$.

Proof of (9.2.4): It suffices to prove the inclusion $(C_p \cup \Sigma_q) \subset (\Theta \cap \tau_y(\Theta))$, as the cohomology classes of both sides are equal. The curves C_p and C_q are contained in Θ and $\iota(\Theta) = \Theta$, hence Σ_p and Σ_q are contained in Θ. Now C_q is contained in Θ and so $C_p = \tau_y(C_q)$ is contained in $\tau_y(\Theta)$. In addition, $\iota(\tau_{-y}(\Theta)) = \tau_y(\iota(\Theta)) = \tau_y(\Theta)$, and so

$$\Sigma_q = \iota(C_q) = \iota(\tau_{-y}(C_p)) = \tau_y(\iota(C_p)) = \tau_y(\Sigma_p) \tag{9.2.6}$$

is contained in $\tau_y(\Theta)$.

Proof of (9.2.5):

$$p+r\in\Sigma_q \Leftrightarrow \exists t\in C,\ \omega_C(-q-t)\cong\mathcal{O}_C(p+r)\Leftrightarrow \exists t\in C,\ \omega_C\cong\mathcal{O}_C(p+q+r+t).$$

If $r=t$ the curves C_p and Σ_q are tangent at $\mathcal{O}_C(p+r)$, since their intersection number in Θ is 2.

Step 2: We show next that given $y\in \mathrm{Pic}^0(C)$, the curve C_p is contained in $\tau_y(\Theta)$, if and only if $y=\mathcal{O}_C(p-r)$, for some $r\in C$.

$$C_p\subset\tau_y(\Theta)\Leftrightarrow \exists r\in C,\ y=\mathcal{O}_C(p-r). \tag{9.2.7}$$

The "if" implication is clear. We prove the "only if" implication. Assume that C_p is contained in $\tau_y(\Theta)$. Let L be the line bundle represented by $-y$. Then $\tau_{-y}(C_p)$ is contained in Θ, if and only if the line bundle $\mathcal{O}_C(p+q)\otimes L$ is effective, for all $q\in C$. In this case the point p will belong to the support of the effective divisor in the linear system $|\mathcal{O}_C(p+q)\otimes L|$, for some q as we vary q in C. Hence, if C_p is contained in $\tau_y(\Theta)$, then $L\cong\mathcal{O}_C(r-q)$, for some points $q,r\in C$. Furthermore, $\mathcal{O}_C(r-q+p+q')$ is effective, for all $q'\in C$. So q is a base point of $\mathcal{O}_C(r+p+q')$, for the generic $q'\in C$ for which $h^0(\mathcal{O}_C(r-q+p+q'))=1$ (i.e., for $q'\in C$, such that the images of r,p,q' on the canonical curve are not colinear, or if $p=r$, then for $q'\in C$, such that q' is not on the line tangent to the canonical curve at p). For such generic q' the base locus is $\{r,p,q'\}$. Hence q is equal to one of r or p. If $q=p$ we are done. If $q=r$, then L is trivial and we are done.

Applying ι to (9.2.7) and replacing y by $-y$ we get

$$\Sigma_p\subset\tau_y(\Theta)\Leftrightarrow \exists r\in C,\ y=r-p. \tag{9.2.8}$$

Combing (9.2.7) and (9.2.8) we see that Θ is the only translate of Θ containing both C_p and Σ_p. Indeed, if $p-r=r'-p$, for points $r,r'\in C$, then $2p=r+r'$ and so $p=r=r'$, since C is non-hyperelliptic.

Step 3: Combining the results of the above two steps we conclude that if $\Theta\cap\tau_t(\Theta)=C_p\cup C'$, for some $t\in\mathrm{Pic}^0(C)$ and some curve C', then $t=p-q$, for some $q\in C$, and $C'=\Sigma_q$. Furthermore, the scheme $C_p\cap\Sigma_q$ has length 2.

We prove next that if $C_p\subset\tau_u(\Theta)$, then $\tau_u(\Theta)$ is the unique translate of Θ containing $C_p\cup\tau_{2u}(\Sigma_p)$. Indeed, the assumed inclusion implies that $u=p-r$, for some $r\in C$, by (9.2.7). Hence, $\tau_u(\Sigma_p)=\Sigma_r\subset\Theta$ and so $\tau_{2u}(\Sigma_p)\subset\tau_u(\Theta)$. Assume that $C_p\cup\tau_{2u}(\Sigma_p)\subset\tau_u(\Theta)\cap\tau_t(\Theta)$. Applying τ_{-u} to both sides we get the inclusion $C_r\cup\Sigma_r\subset\Theta\cap\tau_{t-u}(\Theta)$. Thus, $t-u=r-r=0$, by the previous paragraph. So $t=u$.

Step 4: We show next that if $C_p\cup\tau_s(\Sigma_p)=\tau_{t_1}(\Theta)\cap\tau_{t_2}(\Theta)$, for some $t_1,t_2\in\mathrm{Pic}^0(C)$, then

1. $t_1=p-q_1$ and $t_2=p-q_2$, for some $q_1,q_2\in C$,
2. $s=2p-q_1-q_2$ and $s\neq 0$,
3. $C_p\cap\tau_s(\Sigma_p)=\{p+r,p+t\}$, where $\mathcal{O}_C(q_1+q_2+r+t)\cong\omega_C$. Furthermore, C_p is tangent to $\tau_s(\Sigma_p)$ at $p+r$, if $r=t$.

This proves the uniqueness of the pair of translates of Θ whose complete intersection is $C_p\cup\tau_s(\Sigma_p)$ in Part (3a).

(1) follows from Equation (9.2.7).

(2) Applying τ_{-t_1} we get that $C_{q_1} \cup \tau_{s-t_1}(\Sigma_p) = \Theta \cap \tau_{q_1-q_2}(\Theta)$. So $\tau_{s-t_1}(\Sigma_p) = \Sigma_{q_2}$, by (9.2.4) and $s - t_1 = p - q_2$, by (9.2.6). This proves the equality in (2). If $s = 0$, then $p = q_1 = q_2$ and so $t_1 = t_2$, which contradicts the assumed equality $C_p \cup \tau_s(\Sigma_p) = \tau_{t_1}(\Theta) \cap \tau_{t_2}(\Theta)$.

(3) We have $C_{q_1} \cap \tau_{s-t_1}(\Sigma_p) = C_{q_1} \cap \tau_{p-q_2}(\Sigma_p) = C_{q_1} \cap \Sigma_{q_2} = \{q_1 + r, q_1 + t\}$, where $\mathcal{O}_C(q_1+q_2+r+t) \cong \omega_C$. The first equality follows from (1) and (2), the second equality from (9.2.6), and the third equality from Step 1. If $r = t$ the two curves C_{q_1} and Σ_{q_2} are tangent at $q_1 + r$, by Step 1. Hence, C_p is tangent to $\tau_s(\Sigma_p)$ at $p + r$.

Step 5: If $C_p \cap \tau_s(\Sigma_p)$ is non-empty, then $C_p \cup \tau_s(\Sigma_p)$ is contained in a translate of Θ, by Lemma 9.2.5. It remains to prove that in this case the length of the subscheme $C_p \cap \tau_s(\Sigma_p)$ is 2. As in the proof of Step 4(3) we can translate so that the subscheme in question is $C_{q_1} \cap \Sigma_{q_2}$ for two points $q_1, q_2 \in C$. Now the intersection number of these two curves in Θ is 2. This completes the proof of Parts (1) and (2). □

Lemma 9.2.5. *Let t be a point of $\mathrm{Pic}^0(C)$. If $C_p \cap \tau_t(\Sigma_p)$ is non-empty, then $2p-t \sim a+b$, for a unique effective divisor $a+b$, $a, b \in C$. In that case the union $Z_{p,t} := C_p \cup \tau_t(\Sigma_p)$ is contained in each of $\tau_{p-a}(\Theta)$ and $\tau_{p-b}(\Theta)$.*

Proof. Step 1: Assume that $C_p \cap \tau_t(\Sigma_p) \neq \emptyset$. Then t belongs to the surface $C_p - \Sigma_p$ in $\mathrm{Pic}^0(C)$, i.e., $t = p+q-r-s$, where $r+s$ belongs to Σ_p. The fact that the divisor $r+s$ belongs to Σ_p means that there exists a point $u \in C$, such that $\mathcal{O}_C(p+u+r+s) \cong \omega_C$.

Step 2: We show next that there exists a unique effective divisor $a + b$, $a, b \in C$, such that $a + b + q \sim p + r + s$. The dimension of $H^0(C, \mathcal{O}_C(p + r + s))$ is 2, by Step 1. So $H^0(C, \mathcal{O}_C(p + r + s - q))$ is 1-dimensional, since C is not hyperelliptic. Hence, $p + r + s - q \sim a + b$, for a unique effective divisor $a + b$, $a, b \in C$.

Step 3: We have

$$\tau_{-t}(\tau_{p-a}(\Sigma_b)) = \tau_{r+s-q-a}(\Sigma_b) \overset{(9.2.6)}{=} \tau_{r+s-q-a}(\tau_{p-b}(\Sigma_p)) = \tau_{r+s+p-q-a-b}(\Sigma_p) = \Sigma_p,$$

where the last equality follows from Step 2. We conclude the equality $\tau_t(\Sigma_p) = \tau_{p-a}(\Sigma_b)$. Hence,

$$C_p \cup \tau_t(\Sigma_p) = \tau_{p-a}(C_a) \cup \tau_{p-a}(\Sigma_b) \subset \tau_{p-a}(\Theta).$$

Interchanging the roles of a and b we get also that $C_p \cup \tau_t(\Sigma_p)$ is contained in $\tau_{p-b}(\Theta)$. Finally we have

$$a + b \sim p - q + r + s \sim (p - q) + (p + q - t) = 2p - t.$$

Uniqueness of $a + b$ follows, since C is assumed non-hyperelliptic. □

Remark 9.2.6. Keep the notation of Lemma 9.2.5. If $C_p \cap \tau_t(\Sigma_p)$ is non-empty, then the canonical line bundle $\omega_{Z_{p,t}}$ is the restriction of $\mathcal{O}_X(\tau_{p-a}(\Theta) + \tau_{p-b}(\Theta))$, by Lemma 9.2.4(3). Hence $\omega_{Z_{p,t}}$ is isomorphic to $\mathcal{O}_X(3\Theta - \tau_{-t}(\Theta))$, by the Theorem of the Square.

As a corollary, we get the following.

Lemma 9.2.7. *Let s and t be points of $\mathrm{Pic}^0(C)$. If $\tau_s(C_p) \cap \tau_t(\Sigma_p)$ is non-empty, then the following statements hold.*

(1) *$s - t + 2p \sim a + b$, $a, b \in C$, for a unique effective divisor $a + b$.*

(2) *The union* $Z_{p,s,t} := \tau_s(C_p) \cup \tau_t(\Sigma_p)$ *is contained in each of* $\tau_{p+s-a}(\Theta)$ *and* $\tau_{p+s-b}(\Theta)$ *and its canonical line bundle* $\omega_{Z_{p,s,t}}$ *is the restriction of* $\mathcal{L}_{s,t} := \mathcal{O}_X(3\Theta - \tau_{-t-s}(\Theta))$.

(3) *Let* $D_{p,s,t}$ *be the degree 2 divisor on* $\tau_s(C_p)$ *corresponding to the length 2 subscheme* $\tau_s(C_p) \cap \tau_t(\Sigma_p)$. *Denote by* $D'_{p,s,t}$ *the analogous degree 2 divisor on* $\tau_t(\Sigma_p)$. *The line bundle* $\mathcal{L}_{s,t}$ *is the unique line bundle in its connected component of* $\mathrm{Pic}(X)$, *which restriction to* $\tau_s(C_p)$ *is* $\omega_{\tau_s(C_p)}(D_{p,s,t})$. *The line bundle* $\mathcal{L}_{s,t}$ *is also the unique line bundle in its connected component of* $\mathrm{Pic}(X)$, *which restriction to* $\tau_t(\Sigma_p)$ *is* $\omega_{\tau_t(\Sigma_p)}(D'_{p,s,t})$.

Proof. Part (1) follows from Lemma 9.2.5.

Part (2): The statement is a translate by τ_s of that of Lemma 9.2.5 and Remark 9.2.6. The canonical line bundle $\omega_{Z_{p,s,t}}$ is thus the restriction of $\mathcal{O}_X(\tau_{p+s-a}(\Theta) + \tau_{p+s-b}(\Theta))$. The latter is linearly equivalent to $3\Theta - \tau_{-t-s}(\Theta)$, by the Theorem of the Square.

Part (3): $\mathcal{L}_{s,t}$ restricts to $Z_{p,s,t}$ as the canonical line bundle $\omega_{Z_{p,s,t}}$, by part (2), and the latter restricts to $\tau_s(C_p)$ as $\omega_{\tau_s(C_p)}(D_{p,s,t})$. If $\mathcal{L}'$ is another translate of $\mathcal{O}_X(2\Theta)$, which restricts to $\tau_s(C_p)$ as $\omega_{\tau_s(C_p)}(D_{p,s,t})$, then $\mathcal{L}'$ is isomorphic to $\mathcal{L}_{s,t}$, since the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(\tau_s(C_p))$ induces an isomorphism from each connected component of $\mathrm{Pic}(X)$ onto the corresponding connected component of $\mathrm{Pic}(\tau_s(C_p))$. The same argument proves the analogous statement for $\tau_t(\Sigma_p)$. □

Proof of Proposition 9.2.2. Reduction of part (2) to part (1). The following tensor products are in the derived category. The object $\pi_1^*F_1 \otimes \mathcal{F}_2$ in $D^b(X \times X \times \hat{X})$ is the tensor product of the line bundle $\pi_1^*(\mathcal{O}_X(\Theta)) \otimes \pi_{12}^*(a^*(\mathcal{O}_X(\Theta))) \otimes \pi_{13}^*\mathcal{P}^{-1}$ and the object $\pi_1^*(\mathcal{I}_{\cup_{i=1}^{d+1} C_i}) \otimes \pi_{12}^*(a^*(\mathcal{I}_{\cup_{j=1}^{d+1}\Sigma_j}))$. Each of the factors in the latter tensor product is the ideal sheaf of a subscheme flat over $X \times \hat{X}$ with respect to π_{23}. The object $\pi_1^*(\mathcal{I}_{\cup_{i=1}^{d+1} C_i}) \otimes \pi_{12}^*(a^*(\mathcal{I}_{\cup_{j=1}^{d+1}\Sigma_j}))$ restricts to the fiber of π_{23} over $(x_2, L) \in X \times \hat{X}$ as the object $\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{I}_{\cup_{j=1}^{d+1}\tau_{-x_2}(\Sigma_j)}$, which is isomorphic to the ideal sheaf of a subscheme of X, by Lemma 11.0.1 (the torsion sheaves $\mathcal{T}or_k\left(\mathcal{I}_{\cup_{i=1}^{d+1} C_i}, \mathcal{I}_{\cup_{j=1}^{d+1}\tau_{-x_2}(\Sigma_j)}\right)$ vanish, for $k \neq 0$). Hence, the object $\pi_1^*(\mathcal{I}_{\cup_{i=1}^{d+1} C_i}) \otimes \pi_{12}^*(a^*(\mathcal{I}_{\cup_{j=1}^{d+1}\Sigma_j}))$ is isomorphic to a coherent sheaf over $X \times X \times \hat{X}$, which is flat over $X \times \hat{X}$ with respect to π_{23}. Consequently, so is $\pi_1^*F_1 \otimes \mathcal{F}_2$.

There exists over $X \times \hat{X}$ a complex of locally free sheaves of finite rank

$$K^\bullet : K^0 \stackrel{d_0}{\to} \cdots \stackrel{d_{p-1}}{\to} K^p \stackrel{d_p}{\to} K^{p+1} \stackrel{d_{p+1}}{\to} \cdots \stackrel{d_{n-1}}{\to} K^n$$

representing the object $\mathcal{G} := R\pi_{23,*}(\pi_1^*F_1 \otimes \mathcal{F}_2)$ in $D^b(X \times \hat{X})$. Furthermore, for every subscheme B of $X \times \hat{X}$ we have

$$R\pi_{23,*}(\pi_1^*F_1 \otimes \mathcal{F}_2 \otimes \pi_{23}^*\mathcal{O}_B) \cong (K^\bullet) \otimes \mathcal{O}_B,$$

by cohomology and base change (see the theorem in section 5 of [Mum]).

The restriction of $\mathcal{G}$ to the fiber $\pi_{23}^{-1}((x, L))$ is $F_1 \otimes \tau_x^*(F_2) \otimes L^{-1}$. We will prove part (1) by showing that

(i) $H^i(X, F_1 \otimes \tau_x^*(F_2) \otimes L^{-1})$ vanishes for $i \neq 1$ for all $(x, L) \in [X \times \mathrm{Pic}^0(X)] \setminus \tilde{\Theta}$.

(ii) For $(x, L) \in \tilde{\Theta}$ the cohomology $H^i(X, F_1 \otimes \tau_x^*(F_2) \otimes L^{-1})$ vanishes if and only if $i \notin \{1, 2\}$.

It follows that d_0 is fiberwise injective and thus $K^1/Im(d_0)$ is locally free. We may thus assume that $K^0=0$. Furthermore, if $n>2$, then the rightmost homomorphism $d_{n-1}:K^{n-1}\rightarrow K^n$ is fiberwise surjective, by the vanishing of $H^n(X,F_1\otimes\tau_x^*(F_2)\otimes L^{-1})$. Hence, $\ker(d_{n-1})$ is locally free. We may thus assume that $K^p=0$, for $p>2$. The complex is thus $K^1\stackrel{d_1}{\rightarrow}K^2$ and the cokernel of d_1 is supported, set theoretically, on $\tilde{\Theta}$. Hence, the object $\mathcal{G}^\vee[-1]$ is represented by the complex $(K^\bullet)^*[-1]:=(K^2)^*\stackrel{d_1^*}{\rightarrow}(K^1)^*$, where $(K^1)^*$ is in degree 0 and d_1^* is an injective sheaf homomorphism, whose cokernel $\mathcal{E}$ has rank equal to the rank $8d$ of the kernel of d_1.

Applying $\mathcal{RH}om(\bullet,\mathcal{O}_{X\times\hat{X}})$ to the short exact sequence

$$0\rightarrow(K^2)^*\stackrel{d_1^*}{\rightarrow}(K^1)^*\rightarrow\mathcal{E}\rightarrow 0$$

we get the exact sequence

$$0\rightarrow\mathcal{G}_1\rightarrow K^1\stackrel{d_1}{\rightarrow}K^2\rightarrow\mathcal{E}xt^1(\mathcal{E},\mathcal{O}_{X\times\hat{X}})\rightarrow 0.$$

We conclude that $\mathcal{G}_1\cong\mathcal{E}^*$, $\mathcal{G}_2\cong\mathcal{E}xt^1(\mathcal{E},\mathcal{O}_{X\times\hat{X}})$, and $\mathcal{E}xt^i(\mathcal{E},\mathcal{O}_{X\times\hat{X}})=0$, for $i>2$. The sheaf $\mathcal{G}_1$ is a saturated subsheaf of K^1, and is thus reflexive. The codimension of the support of $\mathcal{E}xt^1(\mathcal{E},\mathcal{O}_{X\times\hat{X}})$ is 4. We see that if $\mathcal{E}xt^i(\mathcal{E},\mathcal{O}_{X\times\hat{X}})$ does not vanish, then the codimension of its support is larger than $i+1$, for all $i>0$. Hence $\mathcal{E}$ is reflexive, by [HL, Prop. 1.1.10(3')].

Proof of part (1). It suffices to prove (i) and (ii) above. The tensor product $F_1\otimes\tau_x^*F_2$ is isomorphic to $\mathcal{I}_{Z_x}(\Theta+\tau_{-x}(\Theta))$ for a subscheme Z_x of X supported on the union of the curves C_i and $\tau_{-x}(\Sigma_j)$, $1\leq i,j\leq d+1$, possibly with embedded points at points of intersections of C_i and $\Sigma'_j:=\tau_{-x}(\Sigma_j)$, by Lemma 11.0.1. Set $L':=L^{-1}(\tau_{-x}(\Theta)-\Theta)$. We have the short exact sequence

$$0\rightarrow F_1\otimes\tau_x^*(F_2)\otimes L^{-1}\rightarrow\mathcal{O}_X(2\Theta)\otimes L'\rightarrow\mathcal{O}_{Z_x}(2\Theta)\otimes L'\rightarrow 0. \tag{9.2.9}$$

$H^i(\mathcal{O}_X(2\Theta)\otimes L')$ vanishes, for $i>0$, and $H^0(\mathcal{O}_X(2\Theta)\otimes L')$ is 8-dimensional. We have $\chi(\mathcal{O}_{Z_x}(2\Theta)\otimes L')=8(d+1)$. The later statement is clear when the curves C_i and Σ'_j are disjoint for all $1\leq i,j\leq d+1$. In this case Z_x is the disjoint union of these $2d+2$ curves,

$$\mathcal{O}_{Z_x}(2\Theta)\otimes L'\cong\left[\left(\oplus_{i=1}^{d+1}\mathcal{O}_{C_i}\right)\oplus\left(\oplus_{i=1}^{d+1}\mathcal{O}_{\Sigma'_i}\right)\right]\otimes\mathcal{O}_X(2\Theta)\otimes L',$$

The restriction of $\mathcal{O}_X(2\Theta)\otimes L'$ to each of C_i and Σ'_j has degree 6 and thus Euler characteristic 4. Hence, $\chi(\mathcal{O}_Z(2\Theta)\otimes L')=8(d+1)$ as claimed.

It suffices to prove the two vanishing and one non-vanishing statements below.

$$H^0(F_1\otimes\tau_x^*(F_2)\otimes L^{-1}) = 0, \quad \forall L\in \mathrm{Pic}^0(X), \tag{9.2.10}$$

$$H^1(\mathcal{O}_{Z_x}(2\Theta)\otimes L') = 0, \quad \forall (x,L)\in X\times\hat{X}\setminus\tilde{\Theta}. \tag{9.2.11}$$

$$h^2(F_1\otimes\tau_x^*(F_2)\otimes L) \neq 0, \quad \forall (x,L)\in\tilde{\Theta}. \tag{9.2.12}$$

Statements (i) and (ii) above would follow from the above three statements. The vanishing of $H^i(F_1\otimes\tau_x^*(F_2)\otimes L)$, for $i\geq 2$ and $(x,L)\not\in\tilde{\Theta}$, would then follow from the vanishing (9.2.11) and the long exact sheaf cohomology sequence associated to the short exact sequence (9.2.9). The vanishing for $i=3$ and $(x,L)\in\tilde{\Theta}$ follows from the vanishing of $H^2(\mathcal{O}_{Z_x}(2\Theta)\otimes L')$ and the latter long exact sheaf cohomology sequence.

The vanishing (9.2.10) follows from Assumption 9.1.1, as $\cup_{i=1}^{d+1} C_i$ is a subscheme of Z_x.

The vanishing (9.2.11) is clear, when the curves C_i and Σ'_j are disjoint, for $1 \leq i, j \leq d+1$, by Riemann-Roch.

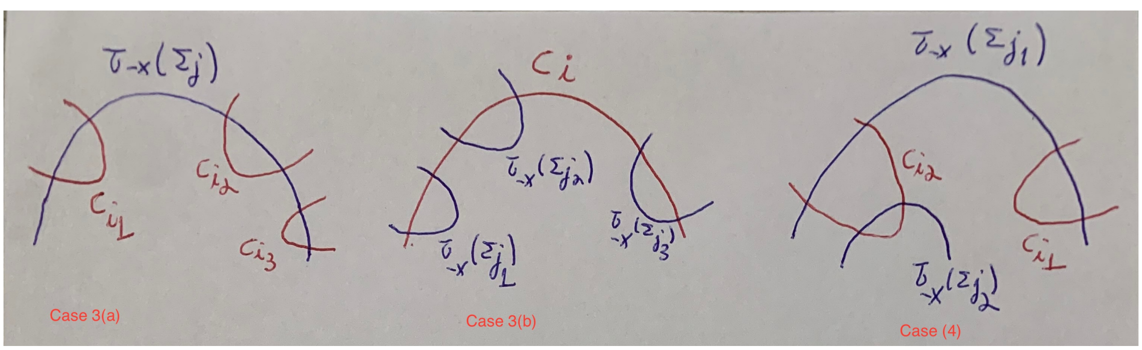
Let $Z_{red} := (\cup_{1 \leq i \leq d+1} C_i) \cup (\cup_{1 \leq j \leq d+1} \Sigma'_j)$ be the reduced induced subscheme of Z_x. We have the short exact sequence

$$0 \to \mathcal{O}_{Z_{tor}}(2\Theta) \otimes L' \to \mathcal{O}_{Z_x}(2\Theta) \otimes L' \to \mathcal{O}_{Z_{red}}(2\Theta) \otimes L' \to 0.$$

The sheaf $\mathcal{O}_{Z_{tor}}(2\Theta) \otimes L$ has zero-dimensional support, hence $H^1(\mathcal{O}_{Z_{tor}}(2\Theta) \otimes L)$ vanishes. The vanishing (9.2.11) would thus follow from the vanishing of $H^1(\mathcal{O}_{Z_{red}}(2\Theta) \otimes L)$.

The locus in X of points x, such that $\tau_{-x}(\Sigma_j)$ and C_i meet is the surface $\Theta_{i,j} = \Sigma_j - C_i$. The connected components of Z_{red} are thus all of the following type, by Lemma 9.2.4 and Assumption 9.2.1(1).

(1) A smooth curve of genus 3.
(2) Case $x \in \Theta_{i,j}$ and $x \notin \Theta_{i',j}$ if $i \neq i'$ and $x \notin \Theta_{i,j'}$ if $j \neq j'$. In that case one connected component is the union of C_i and $\tau_{-x}(\Sigma_j)$ meeting along a length 2 subscheme.
(3) Case (a) $x \in \Theta_{i_1,j} \cap \Theta_{i_2,j} \cap \cdots \cap \Theta_{i_k,j}$, where $k \in \{2,3\}$ and $i_1, \dots, i_k$ are pairwise distinct, or (b) $x \in \Theta_{i,j_1} \cap \Theta_{i,j_2} \cap \cdots \cap \Theta_{i,j_k}$, where $k \in \{2,3\}$ and $i_1, \dots, i_k$ are pairwise distinct.
(4) Case $x \in \Theta_{i_1,j_1} \cap \Theta_{i_2,j_1} \cap \Theta_{i_2,j_2}$, where $i_1 \neq i_2$ and $j_1 \neq j_2$.



The vanishing (9.2.11) in case (1) is clear. In case (2) and $(x, L) \in \tilde{\Theta}_{i,j}$ the line bundle $\mathcal{O}_{Z_{red}}(2\Theta) \otimes L'$ restricts to the canonical line bundle of the connected component $C_i \cup \tau_{-x}(\Sigma_j)$ of Z_{red}, by Lemma 9.2.7. Hence, $H^1(\mathcal{O}_{Z_x}(2\Theta) \otimes L')$ does not vanish in this case. The nonvanishing (9.2.12) follows from the long exact sheaf cohomology sequence associated to the short exact sequence (9.2.9). The nonvanishing (9.2.12) follows for all $(x, L) \in \tilde{\Theta}$, by semi-continuity.

We prove next the vanishing (9.2.11) in cases (3) and (4) when (x, L) does not belong to $\tilde{\Theta}$. Let T be a connected component of Z_{red}. In Case (3)(b) if we remove C_i from T we get the disjoint union of $\tau_{-x}(\Sigma_{j_\ell})$, $1 \leq \ell \leq k$. In Case (3)(a) if we remove $\tau_{-x}(\Sigma_j)$ from T we get the disjoint union of C_{i_ℓ}, $1 \leq \ell \leq k$. In case (4) $C_{i_2} \cup \tau_{-x}(\Sigma_{j_1})$ is a subscheme of the connected component T of Z_{red}, $C_{i_2} \cap \tau_{-x}(\Sigma_{j_1}) \neq \emptyset$, and if we remove C_{i_2} and $\tau_{-x}(\Sigma_{j_1})$ from T we get a disconnected union T'' of C_{i_1} and $\tau_{-x}(\Sigma_{j_2})$. Denote by T' the union of the components removed (in cases 3(a), 3(b), and (4)). We have the short exact sequence

$$0 \to \mathcal{O}_{T''}(-D) \to \mathcal{O}_T \to \mathcal{O}_{T'} \to 0,$$

where the divisor D is associated to the intersection subscheme of the connected components of T' with T''. If (x, L) does not belong to $\tilde{\Theta}$, then $H^1(\mathcal{O}_{T'}(2\Theta)\otimes L')$ vanishes. This is clear if T' is a smooth curve, as the line bundle has degree 6. If $T' = C_{i_2} \cup \tau_{-x}(\Sigma_{j_1})$ the vanishing follows from the fact that (x, L) does not belong to $\tilde{\Theta}_{i_2,j_1}$ and Lemma 9.2.7(2). Indeed, Lemma 9.2.7(2) yields that $\omega_{C_{i_2}\cup\tau_{-x}(\Sigma_{j_1})} = \omega_{\tau_{s_{i_2}}(C_p)\cup\tau_{t_{j_1}-x}(\Sigma_p)}$ is the restriction of $\mathcal{O}_X(3\Theta - \tau_{x-s_{i_2}-t_{j_1}}(\Theta))$, while $(x, L) \notin \tilde{\Theta}_{i_2,j_1}$ yields the difference of the line bundles in the second step below.

$$\begin{aligned}
\mathcal{O}_{C_{i_2}\cup\tau_{-x}(\Sigma_{j_1})}(2\Theta) \otimes L' &\cong \mathcal{O}_{C_{i_2}\cup\tau_{-x}(\Sigma_{j_1})}(\Theta + \tau_{-x}(\Theta)) \otimes L^{-1}\\
&\not\cong \mathcal{O}_{C_{i_2}\cup\tau_{-x}(\Sigma_{j_1})}(2\Theta + \tau_{-x}(\Theta) - \tau_{2x-t_{j_1}-s_{i_2}}(\Theta))\\
&\cong \mathcal{O}_{C_{i_2}\cup\tau_{-x}(\Sigma_{j_1})}(3\Theta - \tau_{x-t_{j_1}-s_{i_2}}(\Theta)).
\end{aligned}$$

The vanishing of $H^1(\mathcal{O}_{T''}(-D) \otimes \mathcal{O}_X(2\Theta) \otimes L')$ is seen as follows. Each connected component of T'' is a smooth genus 3 curve meeting precisely one connected component of T', the divisor D has degree 2 on each connected component of T'', and Lemma 9.2.7(3) implies that $\mathcal{O}_X(3\Theta - \tau_{x-s_{i_2}-t_{j_1}}(\Theta))$ restricts to each component C'' of T'' as $\omega_{C''}(D'')$, where D'' is the part of the divisor D supported on C''. On the other hand, the assumption that $(x, L) \notin \tilde{\Theta}$ implies that $\mathcal{O}_X(2\Theta) \otimes L'$ is not isomorphic to $\mathcal{O}_X(3\Theta - \tau_{x-s_{i_2}-t_{j_1}}(\Theta))$. Hence, the line bundle $\mathcal{O}_{C''}(-D'') \otimes \mathcal{O}_X(2\Theta) \otimes L'$ has degree 4 but is not isomorphic to $\omega_{C''}$.

The vanishing of $H^1(\mathcal{O}_T(2\Theta) \otimes L')$ follows from the vanishings of $H^1(\mathcal{O}_{T'}(2\Theta) \otimes L')$ and $H^1(\mathcal{O}_{T''}(-D) \otimes \mathcal{O}_X(2\Theta) \otimes L')$. We conclude that $H^1(\mathcal{O}_{Z_{red}}(2\Theta) \otimes L')$ vanishes. As observed above, it implies the vanishing (9.2.11).

Cases (3) and (4) lie in the closure of Case (2) and so $H^1(\mathcal{O}_{Z_x}(2\Theta) \otimes L')$ does not vanish in these cases for $(x, L) \in \tilde{\Theta}$ by semi-continuity. $\square$

Let p be a point of C, let $C_p \subset X = \mathrm{Pic}^2(C)$ be the translate of $AJ(C)$ by p, let $t \in \mathrm{Pic}^0(C)$, and set $C_t = \tau_t(C_p)$. Set $\Theta_t := \tau_t(\Theta)$.

Lemma 9.2.8. *The equality* $\dim H^0(X, \mathcal{I}_{C_t}(2\Theta) \otimes L) = 4$ *holds, for all line bundles* $L \in \mathrm{Pic}^0(X)$.

Proof. Let $s \in \mathrm{Pic}^0(C)$ be such that $L(2\Theta)$ is isomorphic to $\mathcal{O}_X(\Theta_t + \Theta_s)$. Set $D := \Theta_s + \Theta_t$. Note the inclusion $C_t \subset \Theta_t$. Denote by $D_{|C_t}$ and $D_{|\Theta_t}$ the restriction of the divisor class. Consider the following diagram with short exact rows and columns.

$$\begin{tikzcd}
\mathcal{O}_X(\Theta_s) \arrow[r, "="] \arrow[d] & \mathcal{O}_X(\Theta_s) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{I}_{C_t}(D) \arrow[r] \arrow[d] & \mathcal{O}_X(D) \arrow[r] \arrow[d] & \mathcal{O}_{C_t}(D_{|C_t}) \arrow[d, "="] \\
\mathcal{O}_{\Theta_t}(D_{|\Theta_t} - C_t) \arrow[r] & \mathcal{O}_{\Theta_t}(D_{|\Theta_t}) \arrow[r] & \mathcal{O}_{C_t}(D_{|C_t})
\end{tikzcd}$$

The degree of $D_{|C_t}$ is 6 and so $h^0(\mathcal{O}_{C_t}(D_{|C_t})) = 4$. The equality $h^0(\mathcal{O}_X(D)) = 8$ implies that $h^0(\mathcal{I}_{C_t}(D))$ is at least 4, by the middle horizontal short exact sequence. Consider the vertical left short exact sequence. The dimensions $h^i(\mathcal{O}_X(\Theta_s))$ are 1 for $i = 0$ and 0 for $i > 0$. Hence, it suffices to prove that $H^0(\mathcal{O}_{\Theta_t}(D_{|\Theta_t} - C_t))$ is 3-dimensional, as it would follow that $h^0(\mathcal{I}_{C_t}(D)) \leq 4$.

The middle vertical short exact sequence implies that $H^0(\mathcal{O}_{\Theta_t}(D_{|\Theta_t}))$ is 7-dimensional, since $h^1(\mathcal{O}_X(\Theta_s)) = 0$. Hence, the equality $h^0(\mathcal{O}_{\Theta_t}(D_{|\Theta_t} - C_t)) = 3$ would follow from the exactness of the bottom horizontal sequence once we prove the vanishing of $h^1(\mathcal{O}_{\Theta_t}(D_{|\Theta_t} - C_t))$. It suffices to prove that $\mathcal{O}_{\Theta_t}(D_{|\Theta_t} - C_t) \otimes \omega_{\Theta_t}^{-1}$ is ample, by Kodaira's vanishing. Now, $\omega_{\Theta_t} \cong \mathcal{O}_X(\Theta_t)_{|\Theta_t}$. It remains to prove the ampleness of $(\Theta_s)_{|\Theta_t} - C_t$. Ampleness depends only on the numerical class and so we may consider the case where $C_t = C_p$, $\Theta_t = \Theta$, $\Theta_s \cap \Theta = C_p + \Sigma_p$. The divisor Θ is ι-symmetric and $\Sigma_p = \iota(C_p)$, so ampleness of Σ_p follows from that of C_p. Ampleness of C_p is seen via the isomorphism of Θ with $C^{(2)}$ and the ampleness of the pullback of C_p to the cartesian square C^2. Indeed, the pullback of C_p to C^2 is $C \times \{p\} \cup \{p\} \times C$, which is ample. □

Remark 9.2.9. (1) The linear systems in the above Lemma all have base loci larger than C_t. Let $r \in \mathrm{Pic}^0(C)$ be such that $L(2\Theta)$ is isomorphic to $\mathcal{O}_X(2\Theta_r)$. Then every divisor in $|L(2\Theta)|$ is ι'-invariant, where $\iota' := \tau_r \iota \tau_{-r}$. Hence, the dimension of $H^0(X, \mathcal{I}_{C_t \cup \iota'(C_t)}(2\Theta) \otimes L)$ is 4 as well.

(2) Reducible divisors in the linear system $|\mathcal{I}_{C_t}(2\Theta) \otimes L|$ are described in [BL, Prop. 11.9.1(a)], when $C_t \cup \iota'(C_t)$ is the complete intersection of two translates of Θ. They are related to trisecants of $\varphi_{L(2\Theta)}(X)$ in [BL, Prop. 11.9.3].

9.3. **A semiregular reflexive secant$^{\boxtimes 2}$-sheaf.** Set $n := d + 1$. Choose $\mathcal{I}_{\cup_{i=1}^n C_i}$ and $\mathcal{I}_{\cup_{i=1}^n \Sigma_i}$ to each be equivariant with respect to a subgroup G_i, $i = 1, 2$, of X of order n, where G_1 permutes the connected components $\{\Sigma_i\}_{i=1}^n$ transitively and G_2 permutes the connected components $\{C_i\}_{i=1}^n$ transitively.

Lemma 9.3.1. *A generic C admits subgroups G_1 and G_2 of $\mathrm{Pic}^0(C)$, such that Assumption 9.2.1 holds for a G_1 orbit $\{C_i\}_{i=1}^n$ of translates of C_p and a G_2 orbit $\{\Sigma_i\}_{i=1}^n$ of translates of Σ_p.*

Proof. Assumption 9.2.1(2) is equivalent to the condition that the intersection $G_1 \cap G_2$ is trivial, which we assume. We have already observed that the surface $\Theta_{i,j} := \Sigma_j - C_i$ in $\mathrm{Pic}^0(C)$ is a translate of the Θ divisor. Hence, it suffices to prove the existence of G_1 and G_2, such that the intersection of any four translates $\tau_{g_1+g_2}(\Theta)$, $(g_1, g_2) \in G_1 \times G_2$, is empty and any three translates have finite intersection. The property is open in moduli, and so it suffices to prove it for the degenerate case, where C is a chain of elliptic curves E_1, E_2, and E_3, as in the proof of Lemma 9.1.4. The curves E_1 and E_2 intersect at a point $\{p_0\}$, the curves E_2 and E_3 intersect at a point p_1, and the divisor Θ is given in (9.1.2). Choose G_1 and G_2 so that $G_1 \cap G_2 = \{0\}$ and for each i the restriction homomorphism $\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(E_i)$ restricts to the subgroup $\langle G_1, G_2 \rangle$ generated by G_1 and G_2 as an injective homomorphism. Each of the three irreducible components D_i of Θ is disjoint from its translate $\tau_{g_1+g_2}(D_i)$, if $(g_1, g_2) \neq (0, 0)$. Hence, the intersection of any four translates is empty and the intersection $\tau_{g_1+g_2}(\Theta) \cap \tau_{g_1'+g_2'}(\Theta) \cap \tau_{g_1''+g_2''}(\Theta)$

of three translates is the union of $\tau_{g_1+g_2}(D_i)\cap\tau_{g_1'+g_2'}(D_j)\cap\tau_{g_1''+g_2''}(D_k)$, with pairwise distinct i,j,k. Each of the latter intersections consists of precisely one point. Hence, all triple intersections $\tau_{g_1+g_2}(\Theta)\cap\tau_{g_1'+g_2'}(\Theta)\cap\tau_{g_1''+g_2''}(\Theta)$ are finite. $\square$

Set $E' := \mathcal{I}_{\cup_{i=1}^n\Sigma_i}\boxtimes\mathcal{I}_{\cup_{i=1}^n C_i}$. The image of the obstruction map

$$ob_{E'} : HT^2(X\times X)\rightarrow \mathrm{Ext}^2(E',E')$$

is $\mathrm{Ext}^2(E',E')^{G_1\times G_2}$. The inclusion $Im(ob_{E'})\subset \mathrm{Ext}^2(E',E')^{G_1\times G_2}$ follows from the G_i-equivariance of $ob_{\mathcal{I}_{\cup_{i=1}^n C_i}}$ and $ob_{\mathcal{I}_{\cup_{i=1}^n \Sigma_i}}$ and the fact that both groups act trivially on $HT^2(X\times X)$. The inclusion $\mathrm{Ext}^2(E',E')^{G_1\times G_2}\subset Im(ob_{E'})$ follows from the surjectivity of $HT^j(X)\rightarrow \mathrm{Ext}^j(\mathcal{I}_{\cup_{i=1}^n C_i},\mathcal{I}_{\cup_{i=1}^n C_i})^{G_1}$, for $j\leq 2$, and the analogous surjectivity for $\mathcal{I}_{\cup_{i=1}^n\Sigma_i}$, which in turn follows from Lemmas 8.2.7 and 8.2.8. The composition

$$\tilde{\Phi} := \Phi\circ([\Theta\boxtimes\Theta]\otimes) = (id\times\Psi_{\mathcal{P}^{-1}[3]})\circ\mu^*\circ([\Theta\boxtimes\Theta]\otimes),$$

of Orlov's derived equivalence Φ with tensorization by the line bundle $\Theta\boxtimes\Theta$, conjugates the subgroup $G_1\times G_2$ of $X\times X$ to a subgroup G of the identity component $X\times\hat{X}\times \mathrm{Pic}^0(X\times\hat{X})$ of the group of autoequivalences of the derived category of $X\times\hat{X}$. The group G is calculated below in Equation (9.3.1).

The sheaf E' admits a natural $G_1\times G_2$ linearization λ' yielding the $G_1\times G_2$-equivariant sheaf (E',λ'). Hence, the image $\mathcal{G} = \Phi(\mathcal{I}_{\cup_{i=1}^n\Sigma_i}(\Theta)\boxtimes\mathcal{I}_{\cup_{i=1}^n C_i}(\Theta))[-3]$ in $D^b(X\times\hat{X})$ admits a G-linearization $\lambda := \tilde{\Phi}(\lambda')$ yielding a G-equivariant object $(\mathcal{G},\lambda)$ with respect to the action of G on $D^b(X\times\hat{X})$.

Lemma 9.3.2. *The image of the obstruction homomorphism $ob_{\mathcal{G}} : HH^2(X\times X)\rightarrow \mathrm{Hom}(\mathcal{G},\mathcal{G}[2])$ is $\mathrm{Hom}(\mathcal{G},\mathcal{G}[2])^G := \mathrm{Hom}((\mathcal{G},\lambda),(\mathcal{G},\lambda)[2])$.*

Proof. The statement follows from the equality, observed above, of the image of $ob_{E'}$ and $\mathrm{Hom}((E',\lambda'),(E',\lambda')[2])^{G_1\times G_2}$. $\square$

The action of G on $HH^*(X\times\hat{X})$ is trivial, so the latter is also the Hochschild cohomology of $D^b_G(X\times\hat{X})$. The semi-regularity map

$$\sigma : \mathrm{Hom}(\mathcal{G},\mathcal{G}[2])\rightarrow\prod_{q=0}^4 H^{q+2}(\Omega^q_{X\times\hat{X}})$$

restricts to an injective homomorphism from $\mathrm{Hom}(\mathcal{G},\mathcal{G}[2])^G$, by Remark 8.2.5, Lemma 8.3.1, and Remark 9.2.3. In that sense $(\mathcal{G},\lambda)$ is semi-regular. The group G is a finite subgroup of $X\times\hat{X}\times\mathrm{Pic}^0(X\times\hat{X})$ and so it deforms with $D^b(X\times\hat{X})$ to an action on every polarized abelian variety in the same connected component. Below we choose, instead, to pass to an action on the derived category induced only by automorphisms of $X\times\hat{X}$.

Let $\bar{G}$ be the projection of G to $X\times\hat{X}$, considered as the group of translation automorphisms of $X\times\hat{X}$.

Lemma 9.3.3. *If the intersection $G_1\cap G_2$ does not contain*[22] *an element of order* 2, *then the projection $p : G\rightarrow\bar{G}$ is an isomorphism.*

[22]This condition is satisfied for G_1 and G_2 as in Lemma 9.3.1, as then $G_1\cap G_2 = \{0\}$.

Proof. Note that elements of the subgroup G of the identity component $X \times \hat{X} \times \mathrm{Pic}^0(X \times \hat{X})$ of $\mathrm{Aut}(D^b(X \times \hat{X}))$ are determined by their action on sky-scraper sheaves and line bundles. Given $x \in X$, denote by τ_x the translation automorphism of X. Set $L_x := \Theta \otimes \tau_{x,*}(\Theta)^{-1}$. The third isomorphism below is due to the Theorem of the Square.

$$L_x \cong \Theta \otimes \tau_{x,*}(\Theta)^{-1} \cong [\tau_{x,*}(\Theta) \otimes \Theta^{-1}]^{-1} \cong \tau_{-x,*}(\Theta) \otimes \Theta^{-1} \cong \tau_x^*(\Theta) \otimes \Theta^{-1} =: \phi_\Theta(x).$$

The autoequivalence $\tau_x^\Theta := (\Theta \otimes) \circ \tau_{x,*} \circ (\Theta^{-1} \otimes)$ of $D^b(X)$ corresponds to the element $(\tau_{x,*}, L_x)$ of $X \times \mathrm{Pic}^0(X)$.

Let $(x_1, x_2) \in G_1 \times G_2$. The compositions $\mu^* \circ (\tau_{x_1}^\Theta \boxtimes \tau_{x_2}^\Theta) \circ \mu_*$ and $\mu_*^{-1} \circ (\tau_{x_1}^\Theta \boxtimes \tau_{x_2}^\Theta) \circ \mu_*$ are equal. We have:

$$\mu^{-1}((\tau_{x_1}, \tau_{x_2})(\mu(x,y))) = \mu^{-1}(x + y + x_1, y + x_2) = (x + x_1 - x_2, y + x_2).$$

Hence, $\mu^* \circ (\tau_{x_1}, \tau_{x_2})_* \circ \mu_* = (\tau_{x_1 - x_2}, \tau_{x_2})_*$.

Given a line bundle M over $X \times X$, we have $\mu^* \circ (M \otimes) \circ \mu_* \cong (\mu^* M \otimes) \circ \mu^* \circ \mu_* \cong (\mu^* M \otimes)$. Now, $\mu^*(\pi_1^* L_{x_1} \otimes \pi_2^* L_{x_2}) \cong a^* L_{x_1} \otimes \pi_2^* L_{x_2}$, where $a : X \times X \to X$ is the addition. Now $a^* L_{x_1} \cong \pi_1^* L_{x_1} \otimes \pi_2^* L_{x_1}$ and $L_{x_1} \otimes L_{x_2} \cong L_{x_1 + x_2}$, by the theorem of the square. We get

$$\mu^* \circ (\tau_{x_1}^\Theta \boxtimes \tau_{x_1}^\Theta) \circ \mu_* \cong ((\pi_1^* L_{x_1} \otimes \pi_2^*(L_{x_1 + x_2})) \otimes) \circ (\tau_{x_1 - x_2}, \tau_{x_2})_*.$$

Let $\mathcal{P}_x$ be the restriction of the Poincaré line bundle to $\{x\} \times \hat{X}$. The autoequivalence $\Psi_{\mathcal{P}^{-1}[3]} \circ \tau_{x,*} \circ \Phi_{\mathcal{P}}$ of $D^b(\hat{X})$ is isomorphic to tensorization by the line bundle $\mathcal{P}_{-x}$, as we have

$$\begin{aligned} \Psi_{\mathcal{P}^{-1}[3]}(\tau_{x,*}(\Phi_{\mathcal{P}}(\mathcal{P}_y^{-1}[3]))) &\cong \Psi_{\mathcal{P}^{-1}[3]}(\tau_{x,*}(\Phi_{\mathcal{P}}(\Psi_{\mathcal{P}^{-1}[3]}(\mathbb{C}_y)))) \cong \Psi_{\mathcal{P}^{-1}[3]}(\tau_{x,*}(\mathbb{C}_y)) \\ &\cong \Psi_{\mathcal{P}^{-1}[3]}(\mathbb{C}_{x+y}) \cong \mathcal{P}_{x+y}^{-1}[3] \cong \mathcal{P}_y^{-1}[3] \otimes \mathcal{P}_x^{-1}. \end{aligned}$$

The autoequivalence $\Psi_{\mathcal{P}^{-1}[3]} \circ (L_x \otimes) \circ \Phi_{\mathcal{P}}$ is translation $\tau_{L_x,*}$ by the point of $\hat{X}$ corresponding to the isomorphism class of L_x, as we have

$$\Psi_{\mathcal{P}^{-1}[3]}((L_x \otimes \Phi_{\mathcal{P}}(\mathbb{C}_{L_y}))) \cong \Psi_{\mathcal{P}^{-1}[3]}((L_x \otimes L_y)) \cong \Psi_{\mathcal{P}^{-1}[3]}(\Phi_{\mathcal{P}}(\mathbb{C}_{L_{x+y}})) \cong \mathbb{C}_{L_{x+y}}.$$

We get the isomorphism

$$\tilde{\Phi} \circ (\tau_{x_1}, \tau_{x_2})_* \circ \tilde{\Phi}^{-1} \cong ((\pi_1^* L_{x_1} \otimes \pi_2^* \mathcal{P}_{-x_2}) \otimes) \circ (\tau_{x_1 - x_2}, \tau_{L_{x_1 + x_2}})_* \tag{9.3.1}$$

The element $(\tau_{x_1 - x_2}, \tau_{L_{x_1 + x_2}})$ of $\bar{G}$ is the identity, if and only if $x_1 = x_2$ and $x_1 + x_2 = 0$, so that x_1 is a point of order 2 of $G_1 \cap G_2$. $\square$

Assume that G_1 and G_2 are chosen as in Lemma 9.3.1. In particular, $G_1 \cap G_2 = \{0\}$.

Lemma 9.3.4. *If d is even, then the divisibility div$(\det(\mathcal{G}))$ is relatively prime to the order $(d+1)^2$ of G.*

Proof. Let $q : G \to \hat{G} \subset \mathrm{Pic}^0(X \times \hat{X})$ be the projection. Both $p : G \to \bar{G}$ and $q : G \to \hat{G}$ are isomorphism. The order $(d+1)^2$ of $\hat{G}$ is relatively prime to the rank $8d$ of $\mathcal{G}$ and so the map $G \to \mathrm{Pic}(X \times \hat{X})$, given by $g \mapsto \det(\mathcal{G} \otimes q(g)) = \det(\mathcal{G}) \otimes q(g)^{\mathrm{rank}(\mathcal{G})}$ is injective. The G-equivariance of $\mathcal{G}$ implies that the map $G \to \mathrm{Pic}(X \times \hat{X})$, given by $g \mapsto \det(\tau_{p(g),*}(\mathcal{G}))$ must be injective as well, since for $g \in G$ we have

$$\begin{aligned} \mathcal{G} &\cong g(\mathcal{G}) \cong \tau_{p(g),*}(\mathcal{G}) \otimes q(g), \\ \det(\mathcal{G}) &\cong \tau_{p(g),*}(\det(\mathcal{G})) \otimes q(g)^{8d}. \end{aligned}$$

Let $\phi_{\det(\mathcal{G})} : X \times \hat{X} \to \mathrm{Pic}^0(X \times \hat{X})$ be the homomorphism sending a point y to $\tau_y^* \det(\mathcal{G}) \otimes \det(\mathcal{G})^{-1}$. We have

$$\phi_{\det(\mathcal{G})}(p(g)) \cong \phi_{\det(\mathcal{G})}(-p(g))^{-1} := [\tau_{p(g),*}(\det(\mathcal{G})) \otimes \det(\mathcal{G})^{-1}]^{-1} \cong q(g)^{8d}.$$

It follows that $\bar{G}$ intersects trivially the kernel of $\phi_{\det(\mathcal{G})}$. Hence, the order of $\bar{G}$ is relatively prime to the divisibility of $\det(\mathcal{G})$. $\square$

Regardless of the parity of d, the intersection $\bar{G} \cap \ker(\phi_{\det(\mathcal{G})})$ consists of the elements of $\bar{G}$ of order dividing $\gcd(d+1, 8)$, by proof of the above lemma.

Given $g \in G$, let $(x_g, L_g) \in (X \times \hat{X}) \times \mathrm{Pic}^0(X \times \hat{X})$ be the point, such that the autoequivalence g of $D^b(X \times \hat{X})$ is isomorphic to $L_g \otimes \tau_{x_g,*}$. The group G has exponent $n = d+1$. We identify G with the corresponding subgroup of $X \times \hat{X} \times \mathrm{Pic}^0(X \times \hat{X})$. This identification provides a choice of an isomorphism $L_g^n \cong \mathcal{O}_{X \times \hat{X}}$, for every $g \in G$, since $g^n = L_g^n \otimes \tau_{nx_g,*}$ is the identity endofunctor and nx_g is the identity element of the group $X \times \hat{X}$, so that $\tau^n_{x_g,*}$ is the identity endofunctor. The rank r of E is $8d$. If d is even, then $\gcd(r, n) = \gcd(8d, d+1) = 1$ and there exists a positive integer a, such that $ar \equiv -1$ modulo n. Set $D := \det(\mathcal{G})$.

Lemma 9.3.5. *If d is even,*[23] *then the object $\mathcal{G} \otimes D^a$ admits a $\bar{G}$-linearization.*

Proof. The linearization isomorphisms $\lambda_g : \mathcal{G} \to \tau_{x_g,*}(\mathcal{G}) \otimes L_g$, $g \in G$, induce the isomorphisms

$$\begin{aligned} \wedge^r \lambda_g : D &\to \tau_{x_g,*}(D) \otimes L_g^r, \\ \lambda_g \otimes (\wedge^r \lambda_g)^a : \mathcal{G} \otimes D^a &\to \tau_{x_g,*}(\mathcal{G} \otimes D^a) \otimes L_g^{ar+1}. \end{aligned}$$

Now, tensorization with L_g^{ar+1} is a power of the identity endofunctor, as noted above. Hence, $\lambda \otimes (\wedge^r \lambda)^a$ is a $\bar{G}$-linearization for $\mathcal{G} \otimes D^a$. $\square$

We proceed without any assumption on the parity of d. Set $U := X \times \hat{X} \setminus \tilde{\Theta}$ and let $\tilde{\iota} : U \to X \times \hat{X}$ be the inclusion. Clearly, U is $\bar{G}$-invariant. The reflexive sheaf $\mathcal{E}$ of Proposition 9.2.2 is locally free over U and we denote by $\mathbb{P}(\tilde{\iota}^* \mathcal{E})$ the projectivization of the restriction of $\mathcal{E}$ to U. Let $\pi : \mathbb{P}(\tilde{\iota}^* \mathcal{E}) \to U$ be the natural projection.

Set $G^\vee := \{(x_g, L_g^{-1}) \ : \ g \in G\}$, so that the functor $L_g^{-1} \otimes \tau_{x_g,*}$ belongs to $G^\vee$, if and only if $L_g \otimes \tau_{x_g,*}$ belongs to G. The G-linearization λ of $\mathcal{G}$ induces a G-linearization on the first sheaf cohomology $\mathcal{G}^1$ of $\mathcal{G}$, which we denote by λ as well. The isomorphisms $\lambda_g : L_g \otimes \tau_{x_g,*}(\mathcal{G}^1) \to \mathcal{G}^1$ yield the isomorphisms $(\lambda_g^\vee)^{-1} : L_g^{-1} \otimes \tau_{x_g,*}(\mathcal{E}) \to \mathcal{E}$, as $\mathcal{E}$ is defined to be $(\mathcal{G}^1)^*$. In particular, we get a $G^\vee$-linearization $\nu := (\lambda^\vee)^{-1}$ of $\mathcal{E}$. The linearization of $\mathcal{E}$ induces a $\bar{G}$-action on $\mathbb{P}(\tilde{\iota}^* \mathcal{E})$, so that π is $\bar{G}$-equivariant. Set

[23] Note that $\mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\sqrt{-4d})$, so the assumption that d is even does not restrict the compex multiplications we can treat.

$Y := [X \times \hat{X}]/\bar{G}$ and let $q : X \times \hat{X} \to Y$ be the quotient morphism. Set $Y^0 := U/\bar{G}$ and let $\iota : Y^0 \to Y$ be the inclusion. Then $\mathbb{P}(\tilde{\iota}^*\mathcal{E})$ descends to a projective bundle B over Y^0. Denote by $\mathcal{A}$ the Azumaya algebra of B. It extends to a reflexive sheaf over Y, by the Main Theorem of [Siu]. We denote the extension to the whole of Y by $\mathcal{A}$ as well. Then $\mathcal{A}$ is a coherent sheaf of associative algebras with a unit over Y, locally isomorphic to the sheaf of endomorphisms of a reflexive sheaf, and $q^*\mathcal{A}$ is naturally isomorphic to $\mathcal{E}nd(\mathcal{E})$. Let $\mathcal{A}_0$ be the kernel of the trace homomorphism $tr : \mathcal{A} \to \mathcal{O}_Y$. We have the direct sum decomposition $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{O}_Y$, where the second direct summand is generated by the unit. Set $r := \operatorname{rank}(\mathcal{E})$.

Lemma 9.3.6. *There exists a reflexive sheaf $\mathcal{B}$ over Y twisted by a 2-cocycle with coefficients in μ_r and with trivial determinant, which is locally free over Y_0 and such that $\mathbb{P}(\iota^*\mathcal{B}) \cong B$.*

Proof. Step 1: We first construct a twisted reflexive sheaf $\mathcal{B}'$ which is locally free over Y_0 and such that $\mathbb{P}(\iota^*\mathcal{B}') \cong B$. Choose a covering $\bar{\mathcal{U}} := \{\bar{U}_j\}_{j\in\bar{I}}$ of Y, open in the analytic topology and consisting of subsets biholomorphic to open polydiscs, such that $q^{-1}(\bar{U}_j)$ is a disjoint union of open subsets to each of which q restricts as an injective map, and such that if U_i is a connected component of $q^{-1}(\bar{U}_{\bar{i}})$ and U_j is a connected component of $q^{-1}(\bar{U}_{\bar{j}})$ and $\bar{U}_{\bar{i}} \cap \bar{U}_{\bar{j}} \neq \emptyset$, then there exists a unique element $g_{ij} \in \bar{G}$, such that $\tau_{g_{ij}}(U_j)$ intersects U_i. Let L_{ij} be the unique line bundle, such that (g_{ij}, L_{ij}) is an element of $G^\vee$ corresponding to the autoequivalence $L_{ij} \otimes \tau_{g_{ij},*}$. Denote by $\mathcal{U} := \{U_i\}_{i\in I}$ the open covering of $X \times X$ obtained by the connected components of $q^{-1}(U_j)$, $j \in \bar{I}$. Given $i \in I$, denote by $\bar{i} \in \bar{I}$ the index of $q(U_i)$. Let $\mathcal{V} := \{V_i\}_{i\in I}$ be the open covering of Y with $V_i = \bar{U}_{\bar{i}}$. Note that the restriction $q_i : U_i \to V_i$ of q is biholomorphic. If non-empty, the intersection $V_{ij} := V_i \cap V_j$ is the isomorphic image of $U_i \cap \tau_{g_{ij}}(U_j)$ via q.

Let $\mathcal{E}_i$ be the restriction of $\mathcal{E}$ to U_i. Set $\mathcal{B}'_i := q_{i,*}(\mathcal{E}_i)$. If $V_{ij} \neq \emptyset$, let $\rho_{ij} : \mathcal{B}'_{j_{|V_{ij}}} \to \mathcal{B}'_{i_{|V_{ij}}}$ be the composition

$$\mathcal{B}'_{j_{|V_{ij}}} = q_{j,*}(\mathcal{E}_j)_{|V_{ij}} \stackrel{q_*(\bar{\lambda}_{ij})}{\to} q_{j,*}(\tau_{g_{ji},*}(\mathcal{E}_i))_{|V_{ij}} = q_{i,*}(\mathcal{E}_i)_{|V_{ij}} = \mathcal{B}'_{i_{|V_{ij}}},$$

where $\bar{\lambda}_{ij} : \mathcal{E}_{|U_j\cap\tau_{g_{ji}}(U_i)} \to (\tau_{g_{ji},*}\mathcal{E})_{|U_j\cap\tau_{g_{ji}}(U_i)}$ is induced by the isomorphism $\nu_{g_{ji}} : \mathcal{E} \to (\tau_{g_{ji},*}\mathcal{E}) \otimes L_{ji}$ of the $G^\vee$-linearization of $\mathcal{E}$ and by a choice of a trivialization of L_{ji} over U_j. The sheaves $\mathcal{B}'_i$ are clearly locally free over $V_i \cap Y_0$. If $V_{ijk} := V_i \cap V_j \cap V_k$ is non-empty, then the composition $\rho_{ij}\rho_{jk}\rho_{ki}$ is an invertible holomorphic function times the identity automorphism, since $g_{ij}g_{jk}g_{ki}$ is the identity in $\bar{G}$ and so $\bar{\lambda}_{ij} \circ \tau^*_{g_{ij}}(\bar{\lambda}_{jk}) \circ \tau^*_{g_{ik}}(\bar{\lambda}_{ki})$ is an invertible holomorphic function times the identity automorphism. The twisted sheaf $\mathcal{B}' := \{\mathcal{B}'_i, \rho_{ij}\}_{i,j\in I}$ is locally free over Y_0 and the projectivisation of $\iota^*\mathcal{B}'$ is naturally isomorphic to B.

Step 2: We change the gluing transformations of the twisted sheaf $\mathcal{B}'$ by a 1-cochain with coefficients in $\mathcal{O}_Y^\times$ to obtain a twisted sheaf $\mathcal{B}$ twisted by a 2-cocycle with coefficients in μ_r. Choose trivializations $\psi_i : \det(\mathcal{B}'_i) \to \mathcal{O}_{U_i}$. Note that the determinant line bundle is defined for every complex of coherent sheaves, and for our reflexive sheaves $\det(\mathcal{B}'_i)$ is simply the pushforward of $\wedge^r(\mathcal{B}'_{i_{|V_i\cap Y_0}})$ from $V_i \cap Y_0$ to V_i. We get the holomorphic functions $\eta_{ij} := \psi_i \circ \det(\rho_{ij}) \circ \psi_j^{-1}$ over V_{ij}. We have

$$\begin{aligned}\eta_{ij}\eta_{jk}\eta_{ki} &= \psi_i \circ \det(\rho_{ij})\psi_j^{-1}\psi_j \circ \det(\rho_{jk})\psi_k^{-1}\psi_k \circ \det(\rho_{ki})\psi_i^{-1} \\ &= \psi_i \det(\rho_{ij}\rho_{jk}\rho_{ki})\psi_i^{-1} = \det(\rho_{ij}\rho_{jk}\rho_{ki}).\end{aligned}$$

Choose r-th roots $\tilde{\eta}_{ij}$ of η_{ij} (here we further assume that the open covering $\bar{\mathcal{U}}$ was chosen so that $\bar{U}_{ij}$ are simply connected or empty). Set $\rho''_{ij} := \tilde{\eta}_{ij}^{-1}\rho_{ij}$. Then $\rho''_{ij}\rho''_{jk}\rho''_{ki} = (\tilde{\eta}_{ij}\tilde{\eta}_{jk}\tilde{\eta}_{ki})^{-1}\rho_{ij}\rho_{jk}\rho_{ki}$ is again a holomorphic invertible function times the identity automorphism, and

(9.3.2)

$$\det(\rho''_{ij})\det(\rho''_{jk})\det(\rho''_{ki}) = (\tilde{\eta}_{ij}\tilde{\eta}_{jk}\tilde{\eta}_{ki})^{-r}\det(\rho_{ij}\rho_{jk}\rho_{ki}) = (\eta_{ij}\eta_{jk}\eta_{ki})^{-1}\det(\rho_{ij}\rho_{jk}\rho_{ki}) = 1.$$

Hence, $\rho''_{ij}\rho''_{jk}\rho''_{ki} = \theta_{ijk}\mathbb{1}$, where $\mathbb{1}$ is the identity endomorphism of $\mathcal{B}'_{i_{|V_{ijk}}}$ and the 2-cocycle $\{\theta_{ijk}\}$ has coefficients in μ_r. We set $\mathcal{B} := \{\mathcal{B}'_i, \rho''_{ij}\}_{i,j\in I}$.

Step 3: The gluing transformations of $\det(\mathcal{B}) = \{\wedge^r\mathcal{B}'_i, \det(\rho''_{ij})\}$ satisfy the co-cycle condition, by equation (9.3.2). Hence, $\det(\mathcal{B})$ is an untwisted line bundle. We have the equality $\psi_i \circ \eta_{ij}^{-1}\det(\rho_{ij})) \circ \psi_j^{-1} = 1$, by definition of η_{ij}. Hence, the trivializations ψ_i from Step 2 of the restrictions $\det(\mathcal{B}'_i)$ of $\det(\mathcal{B})$ to V_i glue to an isomorphism $\psi : \det(\mathcal{B}) \to \mathcal{O}_Y$. □

Let $\mathcal{B}$ be a twisted sheaf as in Lemma 9.3.6. We have the equality

$$q^*at_{\mathcal{B}} = at_{\mathcal{E}} - \frac{c_1(\mathcal{E})}{r}\cdot id_{\mathcal{E}}, \tag{9.3.3}$$

by Equation (7.3.5).

Remark 9.3.7. If d is even, then the Brauer class of $\mathcal{B}$ in $H^2(Y, \mathcal{O}_Y^*)$ is trivial, since $(\mathcal{G}\otimes D^a, \lambda\otimes(\wedge^r\lambda)^a)$ descends to an object in $D^b(Y)$, whose derived dual $\bar{\mathcal{E}}$ is represented by an untwisted reflexive sheaf, by Lemma 9.3.5. In this case $\mathcal{B}$ can be chosen to be associated to $\bar{\mathcal{E}}$ via Construction 7.3.3.

Consider the exponential Atiyah class $\exp(at_{\mathcal{B}})$ with graded summands $at_k(\mathcal{B})$ in $\mathrm{Hom}(\mathcal{B}, \mathcal{B}\otimes\Omega^k_Y[k])$. Note that $at_0(\mathcal{B})$ is r times the unit section and $at_1(\mathcal{B}) = at_{\mathcal{B}}$. Note that $\kappa(\mathcal{B})$ is the trace of $\exp(at_{\mathcal{B}})$ and $q^*\kappa(\mathcal{B}) = ch(\mathcal{E})\exp(-c_1(\mathcal{E})/r)$, by equation (9.3.3).

Set $M := X \times \hat{X}$ and $\lambda := c_1(\mathcal{E})/r$. Let $\rfloor\exp(\lambda) : H^1(TM) \to HT^2(M)$ be the composition

$$H^1(TM) \hookrightarrow \begin{array}{c} H^2(\mathcal{O}_M) \\ H^1(TM) \\ H^0(\wedge^2 TM)\end{array} \xrightarrow{\rfloor\left(\begin{array}{ccc} 1 & \lambda & \lambda^2/2 \\ 0 & 1 & \lambda \\ 0 & 0 & 1\end{array}\right)} \begin{array}{c} H^2(\mathcal{O}_M) \\ H^1(TM) \\ H^0(\wedge^2 TM)\end{array} \tag{9.3.4}$$

which has image in $H^2(\mathcal{O}_M) \oplus H^1(TM)$.

Lemma 9.3.8. *The following diagram is commutative*

$$\begin{tikzcd}
HT^2(M) \arrow[r] \arrow[d, "\rfloor\exp(\lambda)"'] & \mathrm{End}(H\Omega^*(M)) \arrow[d, "Ad_{\exp(-\lambda)}"] & \eta \arrow[d, mapsto] \\
HT^2(M) \arrow[r] & \mathrm{End}(H\Omega^*(M)) & \exp(-\lambda)\cup[\eta\circ(\exp(\lambda)\cup(\bullet))]
\end{tikzcd}$$

where the horizontal arrows are defined via the $HT^*(M)$-module structure of $H\Omega^*(M)$ and the right vertical arrow is the conjugation by cup product with $\exp(-\lambda)$.

Proof. An element $\eta\in H^2(\mathcal{O}_M)\subset HT^2(M)$ is mapped to itself by $\rfloor\lambda$ and similarly, its action on $H\Omega^*(M)$ commutes with cup product with $\exp(-\lambda)$ and so it is mapped to itself by $Ad_{\exp(-\lambda)}$. Given $\eta\in H^1(TM)$ and $\gamma\in H\Omega^*(M)$ we have

$$\begin{aligned}
(Ad_{\exp(-\lambda)}(\eta))(\gamma) &= \exp(-\lambda)\cup\{\eta\rfloor(\exp(\lambda)\cup\gamma)\}\\
&= \exp(-\lambda)\cup\left\{\eta\rfloor\left[\gamma+\lambda\cup\gamma+\frac{\lambda^2}{2}\cup\gamma+\frac{\lambda^3}{3!}\cup\gamma\cdots\right]\right\}\\
&= \exp(-\lambda)\cup\{\exp(\lambda)\cup(\eta\rfloor\gamma)+\exp(\lambda)\cup(\eta\rfloor\lambda)\cup\gamma\}\\
&= (\eta\rfloor\lambda)\cup\gamma+(\eta\rfloor\gamma).
\end{aligned}$$

The proof for $\eta\in H^0(\wedge^2 TM)$ is similar. □

Lemma 9.3.9. *The following diagram is commutative.*
(9.3.5)

$$\begin{tikzcd}
HT^2(M) \arrow[rrrr, "ob_{\mathcal{E}}:=\exp(at_{\mathcal{E}})"] \arrow[dddddrr, bend right, "\rfloor ch(\mathcal{E})"'] & & & & \mathrm{Ext}^2(\mathcal{E},\mathcal{E})^{G^\vee} \arrow[dddddll, bend left, "\sigma_{\mathcal{E}}"] \\
& H^1(TM) \arrow[ul, "\rfloor\exp(-\lambda)", "(9.3.4)"'] \arrow[rr, "at_{\mathcal{E}}-\lambda\cdot id_{\mathcal{E}}"] \arrow[dddr, bend right, "\rfloor\kappa(\mathcal{E})"'] & & \mathrm{Ext}^2_0(\mathcal{E},\mathcal{E})^{G^\vee} \arrow[ur, "\subset"'] \arrow[dddl, bend left, "q^*\sigma_{\mathcal{B}}"] & \\
& H^1(TY) \arrow[u, "q^*"'] \arrow[rr, "at_{\mathcal{B}}"] \arrow[dr, "\rfloor\kappa(\mathcal{B})"'] & & H^2(Y,\mathcal{A}_0) \arrow[u, "q^*"] \arrow[dl, "\sigma_{\mathcal{B}}"] & \\
& & \prod_{q=1}^4 H^{q+2}(\Omega^q_Y) \arrow[d, "q^*"'] & & \\
& & \prod_{q=1}^4 H^{q+2}(\Omega^q_M) \arrow[d, "\cup\exp(\lambda)"'] & & \\
& & \prod_{q=0}^4 H^{q+2}(\Omega^q_M) & &
\end{tikzcd}$$

Proof. The image of $ob_{\mathcal{E}}$ in $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ is $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})^{G^\vee}$, by Lemma 9.3.2. The commutativity of the outer triangle of the diagram follows from the commutativity of diagram (8.2.4). If the class λ is the first Chern class of a line bundle L, then the right arrow in (9.3.4) is the action of the derived equivalence of tensorization by L on $HT^2(M)$, by

Lemma 9.3.8. In that case the functor $L^{-1}\otimes(\bullet)$ acts on $HT^2(M)$ as in Lemma 9.3.8, by [CBR], and trivially on $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$, as we identify the latter with $\mathrm{Ext}^2(\mathcal{E}\otimes L^{-1},\mathcal{E}\otimes L^{-1})$, and the functor $L\otimes(\bullet)$ conjugates the Atiyah class $at_{\mathcal{E}\otimes L^{-1}}=at_{\mathcal{E}}-\lambda\cdot id_{\mathcal{E}}$ to $at_{\mathcal{E}}$. The commutativity of the upper trapezoid of Diagram (9.3.5) follows in this case. If λ is not integral, then the commutativity of the upper trapezoid follows from the integral case, as the vanishing of the difference of the two linear homomorphisms in $\mathrm{Hom}(H^1(TM),\mathrm{Ext}^2(\mathcal{E},\mathcal{E}))$ for integral values of λ implies the vanishing of the differences over the Zariski closure of the $\mathrm{Pic}(M)$-orbits in $\mathrm{Hom}(H^1(TM),\mathrm{Ext}^2(\mathcal{E},\mathcal{E}))$.

The cohomological action of $L\otimes(\bullet)$ on $H^*(M,\mathbb{C})$ is multiplication by $\exp(\lambda)$. When $\lambda=c_1(L)$, for some line bundle L, and $\eta\in HT^2(M)$ Lemma 9.3.8 and the equality $ch(\mathcal{E})=\exp(\lambda)\cup\kappa(\mathcal{E})$ yield the equality

$$\exp(-\lambda)\cup(\eta\rfloor ch(\mathcal{E}))=(\eta\rfloor\exp(\lambda))\rfloor\kappa(\mathcal{E}).$$

Substitute $\eta\rfloor\exp(-\lambda)$ for η above and multiply on the left by $\exp(\lambda)$ to get the equivalent equality

$$(\eta\rfloor\exp(-\lambda))\rfloor ch(\mathcal{E})=\exp(\lambda)\cup(\eta\rfloor\kappa(\mathcal{E})).$$

The commutativity of the leftmost square follows in this case. It follows in general, by the argument used above taking the Zariski closure of the $\mathrm{Pic}(M)$-orbit.

The commutativity of the rightmost square follows from Equation (9.3.3). The commutativity of the three inner squares is clear. □

Remark 9.3.10. Note that Diagram (7.2.2) appears as the inner triangle in the above diagram. Lemma 9.3.9 provides an alternative proof for the commutativity of Diagram (7.2.2), as it follows from the commutativity of the outer triangle and the six squares in Diagram (9.3.5), which implies the commutativity of the intermediate and inner triangles (as all arrows labeled q^* are isomorphisms and the bottom vertical arrow is injective).

Lemma 9.3.11. *The twisted reflexive sheaf $\mathcal{B}$ is semiregular.*

Proof. The image of $ob_{\mathcal{E}}$ in $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ is $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})^{G^\vee}$, by Lemma 9.3.2. Consequently, the restriction of $\sigma_{\mathcal{E}}$ to $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})^{G^\vee}$ is injective, by Remark 8.2.5, Lemma 8.3.1(2), and Remark 9.2.3. The injectivity of the restriction of $q^*\sigma_{\mathcal{B}}$ to $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})^{G^\vee}$ follows by the commutativity of the rightmost square of diagram (9.3.5). The injectivity of $\sigma_{\mathcal{B}}$ follows by the commutativity of the square with edges $\sigma_{\mathcal{B}}$ and $q^*\sigma_{\mathcal{B}}$, as the arrows labeled q^* are all isomorphisms. □

Proof of Theorem 1.5.1. The discriminant of the Hermitian form is -1, by Lemma 3.1.3. The algebraicity of $\kappa_3(\mathcal{B})$ follows in a non-empty open subset in moduli from the semiregularity of $\mathcal{B}$, proved in Lemma 9.3.11, and Conjecture 7.3.9, as explained in the paragraph preceding the statement of Theorem 1.5.1. The conjecture was verified in our case of families of abelian varieties in Section 7.4.2. The algebraicity of the Hodge-Weil classes follows from that of $\kappa_3(\mathcal{B})$ and Theorem 1.4.1(4), since given a polarized abelian variety of Weil type (A,η,h) the rational endomorphisms $\eta(K)\subset\mathrm{End}_{\mathbb{Q}}(A)$ act on $H^*(A,\mathbb{Q})$ via algebraic correspondences. The locus in moduli where the Hodge-Weil classes are algebraic is a countable union of closed algebraic susbsets [Vo, Sec. 4.2].

Hence, the locus contains the whole irreducible component of moduli of deformations of $(X \times \hat{X}, \eta, h)$. □

10. Abelian sixfolds of Weil type associated to a coherent sheaf with a positive Igusa invariant on an abelian threefold

We used an ad hoc construction of secant sheaves on abelian 3-folds in Lemma 8.1.1. We provide a more conceptual construction of secant sheaves using results of Igusa. The results of this section are not used in the paper.

10.1. **The Igusa quartic.** Let X be an abelian 3-fold, so that V has rank 12. The group $\mathrm{Spin}(V)$ acts naturally on the coordinate polynomial ring $\mathrm{Sym}((S^+_{\mathbb{Q}})^*)$ of the half-spin representation $S^+_{\mathbb{Q}}$. The ring $\mathrm{Sym}((S^+_{\mathbb{Q}})^*)^{\mathrm{Spin}(V)}$ of invariant polynomials is generated by a single polynomial J of degree 4, by [I, Prop. 3]. The hyperplane $V(J)$ in $\mathbb{P}(S^+_{\mathbb{C}})$ defined by J is the tangential variety of the spinor variety [Ab, Remark 2.1.1]. In other words, $V(J)$ is the union of lines in $\mathbb{P}(S^+_{\mathbb{C}})$ tangent to the spinor variety.

We recall next the explicit formula for Igusa's quartic. Let $\{e_1, \dots, e_6\}$ be a basis of $H^1(X, \mathbb{Z})$ with $\int_X e_1 \wedge e_2 \wedge \cdots \wedge e_6 = 1$. Set $[pt_X] := e_1 \wedge e_2 \wedge \cdots \wedge e_6$. For $1 \leq i < j \leq 6$, set

$$e^*_{ij} := (-1)^{i+j-1} e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_6,$$

so that $e_i \wedge e_j \wedge e^*_{ij} = [pt_X]$. Given an alternating matrix A of rank $2r$, denote by $Pf(A)$ the Pfaffian of A normalized so that $Pf \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} = 1$, where I_r is the $r \times r$ identity matrix. An element $x \in S^+_{\mathbb{Q}}$ is a linear combination

$$x = x_0 + \sum_{i<j} x_{ij} e_i \wedge e_j + \sum_{i<j} y_{ij} e^*_{ij} + y_0 [pt_X]$$

with rational coefficients x_0, x_{ij}, y_{ij}, and y_0. Let X_{ij} be the matrix obtained from (x_{ij}) by crossing out the i-th and j-th rows and columns. Define Y_{ij} similarly in terms of (y_{ij}). Then the Igusa quartic is

(10.1.1)
$$J(x) = x_0 Pf((y_{ij})) + y_0 Pf((x_{ij})) + \sum_{i<j} Pf(X_{ij}) Pf(Y_{ij}) - (1/4)(x_0 y_0 - \sum_{i<j} x_{ij} y_{ij})^2,$$

[I, Prop. 3].

Given a non-zero element $w \in S^+_{\mathbb{C}}$ denote by $[w] \in \mathbb{P}(S^+_{\mathbb{C}})$ the line spanned by w.

Lemma 10.1.1. *For each $[w] \in \mathbb{P}(S^+_{\mathbb{C}}) \setminus V(J)$ there exists a unique plane $P_w \subset S^+_{\mathbb{C}}$, containing w, such that the line $\mathbb{P}(P_w)$ intersects the even spinor variety along two pure spinors corresponding to two transversal maximal isotropic subspaces W_i, $i = 1, 2$, of $V_{\mathbb{C}}$. The homomorphism $\rho : \mathrm{Spin}(V_{\mathbb{C}}) \to SO(V_{\mathbb{C}})$ maps the stabilizer $\mathrm{Spin}(V_{\mathbb{C}})_w$, of an element $w \in S^+_{\mathbb{C}}$ spanning $[w]$, isomorphically onto the image of the embedding*

$$e : SL(W_1) \to SO(V_{\mathbb{C}}) \tag{10.1.2}$$

acting on W_2 via the isomorphism $W_1^* \cong W_2$ induced by the pairing (1.2.2). If w belongs to $S^+_{\mathbb{Q}}$, then the plane P_w is defined over $\mathbb{Q}$.

Proof. The homomorphism ρ restricts to an injective homomorphism from $\mathrm{Spin}(V_{\mathbb{C}})_w$ into $SO(V_{\mathbb{C}})$. Indeed, the kernel of ρ has order 2, generated by $-1 \in C(V_{\mathbb{C}})$, and the latter acts as $-id_{S_{\mathbb{C}}}$ on the spin representation. Hence, the kernel of ρ intersects $\mathrm{Spin}(V_{\mathbb{C}})_w$ trivially.

Let $\tilde{V}(J) \subset S^+_{\mathbb{C}}$ be the cone over $V(J)$. The complement $S^+_{\mathbb{C}} \setminus \tilde{V}(J)$ intersects each fiber of $J : S^+_{\mathbb{C}} \to \mathbb{C}$ in a single $\mathrm{Spin}(V_{\mathbb{C}})$-orbit, by [I, Prop. 3]. Hence, it suffices to prove the statement for one such $w \in S^+_{\mathbb{C}}$ in each fiber. Let $w = 1 + d[pt_X]$, $d \neq 0$, where $[pt_X] \in H^6(X, \mathbb{Z})$ is the class Poincaré dual to a point. Then w belongs to the plane spanned by the two pure spinors 1 and $[pt_X]$ and $J(w) = -(1/4)d^2$. The kernel of $m_1 : V_{\mathbb{C}} \to S^-_{\mathbb{C}}$ is the maximal isotropic subspace $W_1 := H^1(\hat{X}, \mathbb{C})$ and the kernel of $m_{[pt_X]} : V_{\mathbb{C}} \to S^-_{\mathbb{C}}$ is $W_2 := H^1(X, \mathbb{C})$. The description of the stabilizer of w as the embedding (10.1.2) is given in this case in [I, Lemma 2]. Clearly, W_1 and W_2 are the only two maximal isotropic subspaces invariant under $e(SL(W_1))$. Hence, the two lines spanned by 1 and $[pt_X]$ are the only pure spinor lines in $S^+_{\mathbb{C}}$ stabilized under $\mathrm{Spin}(V_{\mathbb{C}})_w$ and the line $\mathbb{P}(\mathrm{span}\{1,\ [pt_X]\})$ is the unique line secant to the spinor variety and passing through w, every point of which is $\mathrm{Spin}(V_{\mathbb{C}})_w$-invariant. We conclude, more generally, that for every w with $[w] \in \mathbb{P}(S^+_{\mathbb{C}}) \setminus V(J)$, the stabilizer $\mathrm{Spin}(V)_w$ fixes precisely two lines of even pure spinors, $[u_1]$, $[u_2]$, and that w belongs to the plane $P_w := \mathrm{span}\{u_1, u_2\}$. Furthermore, setting $W_i := \ker(m_{u_i})$, $i = 1, 2$, the stabilizer $\mathrm{Spin}(V)_w$ is isomorphic to the stated embedding (10.1.2) of $SL(W_1)$.

Assume next that w belongs to $S^+_{\mathbb{Q}}$. Then the stabilizer $\mathrm{Spin}(V_{\mathbb{C}})_w$ is defined over $\mathbb{Q}$. The latter determines P_w. Hence, P_w is defined over $\mathbb{Q}$ as well. □

Remark 10.1.2.
1. For any subfield F of $\mathbb{C}$, the level set $J^{-1}(d) \subset S^+_F$, $d \in F$, consists of a single $\mathrm{Spin}(V_F)$-orbit, by [I, Prop. 3]. Furthermore, if $w = 1 + e^*_{14} + e^*_{25} + de^*_{36}$, $d \in F$, then $J(w) = d$.
2. Note that the secant $\mathbb{P}(P_w)$ to the spinor variety in Lemma 10.1.1 intersects the quartic $V(J)$ along the same two points along which it intersected the spinor variety, each with multiplicity 2, since the even spinor variety is contained in the singular locus of $V(J)$. The subspace P_w is defined over $\mathbb{Q}$, and so the length two subscheme of the intersection of $\mathbb{P}(P_w)$ with the spinor variety is defined over $\mathbb{Q}$.

10.2. **Complex multiplication.** If w belongs to $S^+_{\mathbb{Q}}$, then $d := J(w)$ is a rational number, since J is defined over $\mathbb{Q}$. Set $K := \mathbb{Q}[\sqrt{-d}]$. Assume that $-d$ is not a square of a rational number. Let $\sigma : K \to K$ be the involution in $Gal(K/\mathbb{Q})$. Denote by σ also the induced involution on S^+_K, S^-_K, and V_K. Let $Nm : K \to \mathbb{Q}$ be the norm map $Nm(\lambda) = \lambda\sigma(\lambda)$.

Let $\tilde{O}(V_{\mathbb{Q}})$ be the group of rational similarities of $V_{\mathbb{Q}}$, defined in (2.2.3).

Lemma 10.2.1. *Let $w \in S^+_{\mathbb{Q}}$ be a rational class with $d := J(w) > 0$. Set $K := \mathbb{Q}[\sqrt{-d}]$. Then the two maximal isotropic subspaces W_1 and W_2 of $V_{\mathbb{C}}$ invariant under* $\mathrm{Spin}(V_{\mathbb{Q}})_w$ *are defined over K, but not over $\mathbb{Q}$, and $\sigma(W_1) = W_2$. The centralizer of $\rho(\mathrm{Spin}(V_{\mathbb{Q}})_w)$ in $\tilde{O}(V_{\mathbb{Q}})$ is isomorphic to $K^\times$.*

Proof. There exists an element $g \in \mathrm{Spin}(V_K)$, such that $g(w) = 1 + 2\sqrt{-d}[pt_X]$, by the last paragraph in the proof of [I, Prop. 3]. Then $P_{g(w)} = \mathrm{span}\{1, [pt_X]\}$ and so $W_1 :=$

$g^{-1}(H^1(X,\mathbb{Q}))$ and $W_2 := g^{-1}(H^1(\hat{X},\mathbb{Q}))$ are defined over K and are the two maximal isotropic subspaces of V_K invariant under $\mathrm{Spin}(V_\mathbb{Q})_w$. The vanishing $W_1 \cap W_2 = (0)$ follows from the vanishing $H^1(X,\mathbb{Q}) \cap H^1(\hat{X},\mathbb{Q}) = (0)$ of the intersection in $V_\mathbb{Q}$. The maximal isotropic subspaces W_1 and W_2 are not defined over $\mathbb{Q}$, since otherwise the plane P_w would belong to the $\mathrm{Spin}(V_\mathbb{Q})$ orbit of $\mathrm{span}\{1,[pt_X]\}$ and J would have non-positive values on rational points of P_w, by formula (10.1.1). This would contradict the assumption that $J(w) > 0$. The pair $\{W_1, W_2\}$ is invariant under σ, since P_w is, by Lemma 10.1.1. We conclude that $W_2 = \sigma(W_1)$.

The equality $\mathrm{Spin}(V_\mathbb{Q})_w = \mathrm{Spin}(V_\mathbb{Q})_P$ holds, by Remark 2.2.3. Let $\tilde{e} : K^\times \to GL(V_\mathbb{Q})$ be the homomorphism given in (2.2.4). We have seen that the centralizer of $\rho(\mathrm{Spin}(V_\mathbb{Q})_P)$ in $\tilde{O}(V_\mathbb{Q})$ is isomorphic to $K^\times$ in Lemma 2.2.4. □

Given $w \in S^+_\mathbb{Q}$ as in Lemma 10.2.1 and an orientation of the unique secant P through w (so a choice of one of the two maximal isotropic subspaces W_i, $i = 1, 2$), we get the injective group homomorphism $\tilde{e} : K^\times \to GL(V_\mathbb{Q})$ given in (2.2.4). The image of $\tilde{e}$ is contained in $\tilde{O}(V_\mathbb{Q})$ and is equal to the centralizer of $\rho(\mathrm{Spin}(V_\mathbb{Q})_w)$, by Lemma 2.2.4. When P is spanned by Hodge classes, then $\tilde{e}$ defines a complex multiplication on $X \times \hat{X}$ and a 2-dimensional subspace of Hodge-Weil classes, by Lemma 2.2.6 and Corollary 3.2.2.

Example 10.2.2. Moduli spaces of rank 2 vector bundles with trivial determinant on a principally polarized abelian threefold (X, Θ) were studied in [G]. Gulbrandsen proves the non-emptiness of the moduli space $\mathcal{M}(2, 0, d\Theta^2)$ of slope-stable vector bundles F of rank 2 with $c_1(F) = 0$ and $c_2(F) = d\Theta^2$, for every positive integer d [G, Theorem 2.3]. He also proves that the moduli space with $c_2 = \Theta^2$ contains a Zariski open subset of dimension 13 birational to a $\mathbb{P}^1$ bundle over a finite quotient of $X^2 \times_X X^2 \times_X X^2$, where X^2 is considered a variety over X via the group operation [G, Theorem 6.1]. The Chern character of a vector bundle F in $\mathcal{M}(2, 0, d\Theta^2)$ is $w := 2 - d\Theta^2$. We claim that $J(w) = 16d^3$, so that $K = \mathbb{Q}(\sqrt{-d})$. Indeed, there is a basis $\{e_1, e_2, \ldots, e_6\}$ of $H^1(X,\mathbb{Z})$, such that $\Theta = e_1 \wedge e_4 + e_2 \wedge e_5 + e_3 \wedge e_6$. So

$$\Theta^2 = 2\left[e_1 \wedge e_4 \wedge e_2 \wedge e_5 + e_1 \wedge e_4 \wedge e_3 \wedge e_6 + e_2 \wedge e_5 \wedge e_3 \wedge e_6\right] = (-2)(e^*_{36} + e^*_{25} + e^*_{14}),$$

and $w = 2 + 2d(e^*_{36} + e^*_{25} + e^*_{14})$, so $J(w) = 2^4 d^3 J(1 + (e^*_{36} + e^*_{25} + e^*_{14})) = 16d^3$. Note that $w = 2\alpha$, where α is the class in Lemma 8.1.1.

Example 10.2.3. The Igusa invariant of the Chern character of the ideal sheaf of a length n zero dimensional subscheme is $J(1 - n[pt_X]) = -(1/4)n^2 = -(n/2)^2$, and so $K = \mathbb{Q}$

11. Appendix

Let X be a smooth 3-dimensional variety and C and Σ smooth curves on X, such that the subscheme $C \cap \Sigma$ is zero-dimensional. Let

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \tag{11.0.1}$$

be the short exact sequence of the ideal sheaf of C.

Lemma 11.0.1. (1) *Let $\mathcal{I}_{C,C\cap\Sigma}$ be the ideal sheaf of $C\cap\Sigma$ as a subscheme of C. The following sequence is short exact.*

$$0 \to \mathcal{T}or_{-1}(\mathcal{O}_C,\mathcal{O}_\Sigma) \to \mathcal{O}_C\otimes\mathcal{I}_\Sigma \to \mathcal{I}_{C,C\cap\Sigma} \to 0.$$

In particular, the sheaf $\mathcal{O}_C\otimes\mathcal{I}_\Sigma$ has a 1-dimensional support and a subsheaf with a 0-dimensional support.

(2) *The torsion sheaves $\mathcal{T}or_{-k}(\mathcal{I}_C,\mathcal{I}_\Sigma)$ vanish, for $k>0$, and tensoring (11.0.1) with $\mathcal{I}_\Sigma$ we get the short exact sequence*

$$0 \to \mathcal{I}_C\otimes\mathcal{I}_\Sigma \to \mathcal{I}_\Sigma \to \mathcal{O}_C\otimes\mathcal{I}_\Sigma \to 0. \tag{11.0.2}$$

The lemma implies that $\mathcal{I}_C\otimes\mathcal{I}_\Sigma$ is the ideal sheaf of a subscheme Z with embedded points along $C\cap\Sigma$ whose reduced induced subscheme is $C\cup\Sigma$ and $\mathcal{I}_C\otimes\mathcal{I}_\Sigma$ represents also the left derived tensor product.

Proof. Step 1: We prove part (1) as well as that the sheaf $\mathcal{T}or_{-k}(\mathcal{O}_C,\mathcal{O}_\Sigma)$ vanishes, for $k\geq 2$. Let p be a point of intersection of C and Σ. Let x, y be a regular sequence in the stalk $\mathcal{I}_{C,(p)}$. Consider the Koszul complex resolving the stalk $\mathcal{O}_{C,(p)}$.

$$0 \to \mathcal{O}_{X,(p)} \stackrel{(y,-x)}{\to} \mathcal{O}_{X,(p)}\oplus\mathcal{O}_{X,(p)} \stackrel{(x,y)}{\to} \mathcal{O}_{X,(p)} \to \mathcal{O}_{C,(p)}.$$

Tensoring with $\mathcal{O}_{\Sigma,(p)}$ we get the complex

$$\mathcal{O}_{\Sigma,(p)} \stackrel{(\bar{y},-\bar{x})}{\to} \mathcal{O}_{\Sigma,(p)}\oplus\mathcal{O}_{\Sigma,(p)} \stackrel{(\bar{x},\bar{y})}{\to} \mathcal{O}_{\Sigma,(p)}$$

whose homology sheaves represent the stalk of $\mathcal{T}or_{-k}(\mathcal{O}_C,\mathcal{O}_\Sigma)$. The left arrow is injective, since at least one of the restrictions $\bar{x}$ and $\bar{y}$ of x and y to Σ does not vanish. Hence, $\mathcal{T}or_{-k}(\mathcal{O}_C,\mathcal{O}_\Sigma)$ vanishes, for $k\geq 2$.

Choose next a regular sequence (x',y') in the stalk $\mathcal{I}_{\Sigma,(p)}$ and consider the diagram, whose top row is a locally free resolution.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X,(p)} \arrow[r, "{(y',-x')}"] \arrow[d, "="] & \mathcal{O}_{X,(p)}\oplus\mathcal{O}_{X,(p)} \arrow[r] \arrow[d, "="] & \mathcal{I}_{\Sigma,(p)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{X,(p)} \arrow[r, "{(y',-x')}"] & \mathcal{O}_{X,(p)}\oplus\mathcal{O}_{X,(p)} \arrow[r, "{(x',y')}"] & \mathcal{O}_{X,(p)} &
\end{tikzcd}$$

Tensoring with $\mathcal{O}_{C,(p)}$ we get again that $(\bar{y}',-\bar{x}')$ is injective and its cokernel is the stalk of $\mathcal{I}_\Sigma\otimes\mathcal{O}_C$. We conclude that the stalk at p of $\mathcal{T}or_{-1}(\mathcal{O}_C,\mathcal{O}_\Sigma)$, which is $\ker(\bar{x}',\bar{y}')/Im(\bar{y}',-\bar{x}'))$, embeds as a submodule of the stalk of $\mathcal{I}_\Sigma\otimes\mathcal{O}_C$, which is the cokernel of $(\bar{y}',-\bar{x}')$. This completes the proof of part (1).

Step 2: We prove next the vanishing of $\mathcal{T}or_{-k}(\mathcal{O}_C,\mathcal{I}_\Sigma)$, for $k\geq 1$. The exactness of the sequence (11.0.2) follows. Tensoring (11.0.1) with a coherent sheaf F we get the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathcal{T}or_{-2}(\mathcal{O}_C, F) \\ \arrow[r] & \mathcal{T}or_{-1}(\mathcal{I}_C, F) \arrow[r] & 0 \arrow[r] & \mathcal{T}or_{-1}(\mathcal{O}_C, F) \\ \arrow[r] & \mathcal{I}_C \otimes F \arrow[r] & \mathcal{O}_X \otimes F \arrow[r] & \mathcal{O}_C \otimes F \arrow[r] & 0. \end{tikzcd}$$

We get the isomorphisms $\mathcal{T}or_{-k}(\mathcal{I}_C, F) \to \mathcal{T}or_{-k-1}(\mathcal{O}_C, F)$, for $k \geq 1$, and the analogous isomorphism for Σ.

$$\mathcal{T}or_{-k}(\mathcal{I}_C, F) \stackrel{\cong}{\to} \mathcal{T}or_{-k-1}(\mathcal{O}_C, F), \tag{11.0.3}$$

$$\mathcal{T}or_{-k}(\mathcal{I}_\Sigma, F) \stackrel{\cong}{\to} \mathcal{T}or_{-k-1}(\mathcal{O}_\Sigma, F), \tag{11.0.4}$$

for $k \geq 1$. Hence, we have

$$\mathcal{T}or_{-k}(\mathcal{O}_C, \mathcal{I}_\Sigma) \cong \mathcal{T}or_{-k}(\mathcal{I}_\Sigma, \mathcal{O}_C) \stackrel{(11.0.4)}{\cong} \mathcal{T}or_{-k-1}(\mathcal{O}_\Sigma, \mathcal{O}_C) = 0,$$

for $k \geq 1$, where the right vanishing is by Step 1.

Step 3: The isomorphism (11.0.3) with $F = \mathcal{I}_\Sigma$ yields $\mathcal{T}or_{-k}(\mathcal{I}_C, \mathcal{I}_\Sigma) \cong \mathcal{T}or_{-k-1}(\mathcal{O}_C, \mathcal{I}_\Sigma)$, for $k \geq 1$, and the right hand side vanishes, by Step 2. Hence, $\mathcal{T}or_{-k}(\mathcal{I}_C, \mathcal{I}_\Sigma)$ vanishes, for $k > 0$. □

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