Math 624 Spring 2012: Midterm exam

**Problem 1** Suppose $f : \mathbb{R} \to \mathbb{C}$ is a $C^k$ function (i.e., $k$-times continuously differentiable) and periodic of period $2\pi$. Show that the Fourier coefficients of $f$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx},$$

satisfy the bounds

$$c_n = o(n^{-k})$$

(Recall that we say that $c_n = o(g(n))$ if $\lim_{n \to \infty} \frac{c_n}{g(n)} = 0$.) *Hint:* Induction

**Problem 2** Let $(X, \mathcal{M})$ be a measurable space. Suppose $\{\nu_n\}$ is a sequence of measures on $(X, \mathcal{M})$ which is increasing in the sense that

$$\nu_n(A) \leq \nu_{n+1}(A)$$

for all $n = 0, 1, \cdots$ and all $A \in \mathcal{M}$. Define $\nu$ by

$$\nu(A) = \lim_{n \to \infty} \nu_n(A)$$

Show that $\nu$ is a measure.

**Problem 3** Let $(X, \mathcal{M}, \mu)$ be a finite measure space $\mu(X) < \infty$. Let $\mathcal{F}$ be the set of all complex valued measurable functions on $X$ (finite valued but not necessarily integrable). For $f, g \in \mathcal{F}$ let us define

$$\rho(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} d\mu.$$ 

Prove the following assertions

1. $0 \leq \rho(f, g) < \infty$ and $\rho(f, g) = 0$ if and only if $f = g$ a.e.
2. $\rho(f, g) = \rho(g, f)$
3. $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$
4. If $\{f_n\}$ satisfies $\lim_{m,n \to \infty} \rho(f_n, f_m) = 0$, then there exists a complex-valued measurable function $g$ such that $\lim_{n \to \infty} \rho(f_n, g) = 0$.
5. For a sequence $\{f_n\}$ in $\mathcal{F}$ and $f \in \mathcal{F}$ we have $\lim_{n \to \infty} \rho(f_n, f) = 0$ if and only if $f_n$ converges to $f$ in measure.

Note that this problem shows that the set of measurable functions on a finite metric space can be seen as a complete metric space, and that the metric $\rho$ is the metric of convergence in measure.
Problem 4 Let $\nu$ be a Borel measure on the positive real line $[0, \infty)$ such that

$$\Phi(t) = \nu([0,t))$$

is finite for every $t > 0$.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f$ a nonnegative measurable function. For every $t$ consider the level set

$$S(t) = \{x \in X; f(x) > t\}.$$  

1. Prove that

$$\int_X \Phi(f(x))d\mu = \int_{[0,\infty)} \mu(S(t))d\nu$$

2. Compute this formula for (a) $d\nu = dt$, (b) $d\nu = pt^{p-1}dt$ and (c) $\nu = \delta_{t_0}$ (the delta measure at $t_0$).

Problem 5 Consider the function $g: \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = \begin{cases} 
2 & \text{if } 0 \leq y \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

Let $\mu$ be the measure on $\mathbb{R}^2$ which is absolutely continuous with respect to the Lebesgue measure $m \times m$ on $\mathbb{R}^2$ with Radon Nikodym derivative

$$\frac{d\mu}{d(m \times m)} = g$$

Let $T: \mathbb{R}^2 \to \mathbb{R}$ be the map given by $T(x,y) = x$ and and let $\tau = \mu \circ T^{-1}$ be the measure on $\mathbb{R}$ given by

$$\tau(A) = \mu(T^{-1}(A))$$

Find the Lebesgue decomposition of the Lebesgue measure $m$ on $\mathbb{R}$ with respect to $\tau$, $m = m_{ac} + m_{sing}$ and compute the Radon-Nykodym derivative $\frac{d{m_{ac}}}{d\tau}$.