Math 624: Problem set 4

1. Exercise 2, p.380
2. Exercise 4, p.381.
3. Exercise 8, p. 381
4. Suppose $f$ is a convex function on an open interval $I$. Show that if $[a, b] \subseteq I$ then we have
   
   \[ f(b) - f(a) = \int_{[a,b]} f'(x+) \, dx = \int_{[a,b]} f'(x-) \, dx. \]
5. Prove that $L^\infty(X, M, \mu)$ is a Banach space.
6. Suppose $1 \leq p < q < r \leq \infty$.
   
   (a) Show that $L^p \cap L^r$ is a Banach space with with norm $\|f\| \equiv \|f\|_p + \|f\|_r$ and that the inclusion $L^p \cap L^r \to L^q$ is a continuous map.
   
   (b) Show that $L^p + L^r$ is a Banach space with with norm $\|f\| \equiv \inf \{\|g\|_p + \|h\|_r ; f = g + h\}$ and that the inclusion $L^q \to L^p + L^r \to L^q$ is a continuous map.
7. Let $m$ be Lebesgue measure on $\mathbb{R}^d$.
   
   (a) Show that $L^p(\mathbb{R}^d, m)$ and $l^p$ are separable if $1 \leq p < \infty$.
   
   (b) Show that $L^\infty(\mathbb{R}^d, m)$ and $l^\infty$ are not separable.
8. Generalized Hölder’s inequality. Let $1 \leq p_j \leq \infty$ for $j = 1, \cdots, n$ and suppose
   
   \[ \sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \leq 1. \]
   
   Show that if $f_j \in L^{p_j}$ then $\prod_{j=1}^{n} f_j \in L^{r}$ and
   
   \[ \left\| \prod_{j=1}^{n} f_j \right\|_r \leq \prod_{j=1}^{n} \|f_j\|_{p_j}. \]
9. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, g$ be nonnegative functions. Suppose that $0 < p < 1$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that
   
   \[ \int f \, g \, d\mu \geq \left( \int f^p \, d\mu \right)^{1/p} \left( \int g^q \, d\mu \right)^{1/q}. \]
   
   Hint: use Hölder’s inequality for suitable functions.
10. Let \((X, \mathcal{M}, \mu)\) and suppose that \(f \in L^1(\mu) \cap L^2(\mu)\). Prove that

\[
\lim_{p \to 1^+} \|f\|_p = \|f\|_1.
\]

11. (Hölder’s inequality should be called Roger’s inequality or H"older-Roger’s inequality). Let \(f, g\) be positive measurable functions on a measure space \((X, \mathcal{M}, \mu)\). Let \(0 < t < r < m < \infty\).

(a) Show that if the integrals on the right are finite then the following holds (Roger’s inequality)

\[
\left( \int fg^r d\mu \right)^{\frac{m-t}{r}} \leq \left( \int fg^t d\mu \right)^{\frac{m-r}{t}} \left( \int fg^m d\mu \right)^{\frac{r-t}{t}}
\]

_HINT: Use Hölder’s inequality._

(b) Show conversely how the Hölder inequality follows from the Roger’s inequality.

_HINT: let \(t = 1\) and \(m = 2\).