1. Exercise 21, p. 94

2. Let $X$ be a Banach space equipped with a product $x \times y$. If $X$ is an algebra and if

$$\|x \times y\| \leq \|x\| \|y\| \quad \text{for all} \quad x, y \in X$$

then $X$ is called a Banach algebra.

(a) Show that $L^1(\mathbb{R}^d)$ equipped with the convolution product $f \ast g$ is a complex Banach algebra.

(b) Show that $L^1(\mathbb{R}^d)$ equipped with pointwise multiplication $fg$ is not an algebra.

3. Exercise 23, p. 94 (This shows that the Banach algebra $L^1(\mathbb{R}^d)$ has no unit.)

4. Exercise 24, p. 95

5. Exercise 25, p. 95

6. Consider the following two functions defined on $\mathbb{R}$.

$$h^{(1)}(\xi) = c_1 e^{-\delta |\xi|}.$$

$$h^{(2)}(\xi) = c_2 (1 - \delta |\xi|) \chi_{[-\frac{\pi}{\delta}, \frac{\pi}{\delta}]}. $$

Compute their Fourier transforms $\hat{K}^{(1)}(\xi)(y) = \hat{h}_1(y)$ and $\hat{K}^{(2)}(\xi)(y) = \hat{h}_2(y)$ and show that for suitable choices of $c_1$ and $c_2$ (which you should determine) we have

(i) $\int K^{(j)}(y)dm(y) = 1$

(ii) $\lim_{\delta \to 0} \int_{y \geq |\eta|} K^{(j)}(y) dm(y) = 0$, for any $\eta > 0$.  \hspace{1cm} (1)$

i.e. $K^{(j)}_{\delta}$ are good kernels. ( $K^{(1)}_{\delta}$ is known as Abel’s kernel while $K^{(2)}_{\delta}$ is Fejer’s kernel.

Remark: In proving the Fourier inversion formula we have already encountered the good Kernel $K_{\delta}(y) = \hat{g}(y) = \delta^{-d/2} e^{-\pi |y|^2/\delta}$ (Gauss Kernel) which is the Fourier transform of $g(\xi) = e^{-\pi \delta |\xi|^2}$.

7. Show that if $f \in L^1(\mathbb{R}^d)$ then its Fourier transform $\hat{f}(\xi)$ is uniformly continuous.

8. The Riemann-Lebesgue Lemma states that if $f \in L^1$ then $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$. Prove the statement first for a characteristic functions of rectangle, i.e. $f = \chi_R$ and deduce from this that the statement holds for all $f \in L^1$.

9. Problem 1, p. 95

10. Exercise 4, p. 146

11. Exercise 5, p. 146

12. Exercise 7, p. 147