Math 623: Problem set 6

1. Show that the Cauchy-Schwarz inequality \(|(f, g)| \leq \|f\|\|g\|\) is an equality if and only if \(f = cg\) for some \(c \in \mathbb{C}\).

2. Exercise 4, p. 194

3. (a) Show that neither the inclusion \(L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)\) nor the inclusion \(L^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)\) are valid.

(b) Suppose \(E\) is a set of finite measure. Show then that \(L^2(E) \subset L^1(E)\)

4. Exercise 6, p. 194

5. Consider a vector space (real or complex) \(B\) with a norm \(\| \cdot \|\). We may ask the question whether the norm \(\| \cdot \|\) derive from a scalar product, i.e. is there a scalar product \((\cdot, \cdot)\) on \(B\) such that \((f, f) = \|f\|^2\).

(a) Suppose that \(B\) is a real vector space with norm \(\| \cdot \|\). Prove that the norm is induced by a scalar product if and only if the parallelogram law holds, i.e., we have

\[ \|x + y\|^2 + \|x - y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right] \]

and the scalar product is given by

\[ (x, y) := \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \] (1)

Hint: To show that \((x, y)\) given in eq. (1) is additive in the first variable show first that

\[ 4(u + v, w) + 4(u - v, w) = 8(u, w) \]

Deduce from this that \((x + y, z) = (x, z) + (y, z)\). Prove then that \((\alpha x, y) = \alpha (x, y)\) first for integers \(\alpha\) then for rational \(\alpha\).

Remark: The same result holds for complex scalar products but then the scalar product is given by

\[ (x, y) := \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) \] (2)

The proof is similar but more tedious....

(b) Show that the norm on vector space \(L^1(\mathbb{R}^d)\) does not derive from a scalar product

6. Exercise 9, p. 195
7. Exercise 24, p. 198
8. Exercise 25, p. 198
9. Exercise 28, p. 199
10. Exercise 32, p. 201