Lecture 2: Dominated strategies and their elimination

Let us consider a 2-player game, the players being named Robert (the Row player) and Collin (the Column player). Each of the players has a number of strategies at his disposal. In the examples in lecture 1 the number and the nature of strategies were the same for all players but this does not need to be so in general. We will use the notation s_R for a strategy for Robert and s_C for a strategy. The payoff structure is given in a table, for example,

		Collin		
		A	B	C
Robert	I	0 5	4 5	3 0
	II	4 0	3 0	2 5

where the lower left entry in each box is the payoff for Robert while the upper right entry is for Collin.

It will be useful to introduce two matrices, the matrix P_R which summarizes the payoff of Robert and the matrix P_C which summarizes the payoff of Collin. For the above example we have

$$P_R = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 5 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \end{array} \qquad P_C = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 4 & 3 \\ 4 & 3 & 2 \end{pmatrix} \end{array}$$

Definition 1. (Dominated strategy) *For a player a strategy s is dominated by strategy s' if the payoff for playing strategy s' is strictly greater than the payoff for playing s,* no matter *what the strategies of the opponents are.*

For the row player R the domination between strategies can be seen by comparing the rows of the matrices P_R. The strategy s dominate s' for R if $P_R(s, s_C) < P_R(s', s_C)$ for every s_C, that is every element in the row s of the matrix P_R is smaller than the corresponding entry in the row s':

$$P_R = \begin{array}{c} \\ s \\ \\ s' \\ \\ \end{array} \begin{pmatrix} & \cdots & \cdots & \\ a_1 & a_2 & \cdots & a_n \\ & \cdots & \cdots & \\ b_1 & b_2 & \cdots & b_n \\ & \cdots & \cdots & \end{pmatrix} \qquad a_1 < b_1, a_2 < b_2, \cdots, a_n < b_n$$

For the column player C the domination between strategies can be seen by comparing the columns of of the matrices P_C. The strategy s dominate s' for C if $P_R(s_R, s) < P_R(s_R, s')$ for every s_R that is every element in the column s of the matrix P_C is smaller than the corresponding entry in the column s':

$$P_C = \begin{array}{cccc} & s & & s' \\ \end{array}$$
$$P_C = \left(\begin{array}{cccc} & c_1 & & d_1 \\ \vdots & c_2 & \vdots & d_2 \\ \vdots & \vdots & \vdots & \vdots \\ & c_n & & d_n \end{array} \right) \qquad c_1 < d_1, c_2 < d_2, \cdots, c_n < d_n$$

Example: In our analysis of the prisonner's dilemma we have used the domination of strategy. The payoff matrix for Robert is

$$\begin{array}{l} \text{confess} \\ \text{not confess} \end{array} \begin{pmatrix} -6 & 0 \\ -8 & -1 \end{pmatrix}$$

and clearly the strategy "confess" dominates the strategy "not confess". The situation is the same for both players and both should confess. ■

Principle of elimination of dominated strategies: If the players of the game are rational, then they should never use a dominated strategy since they can do better by picking another strategy, no matter what the other players are doing. In the same vein, if a player player thinks his opponents are rational, he will never assume that his opponent uses a dominated strategy against him. And a rational player knows that his opponent knows that he will never use a dominated strategy, and so on and on .....

The logical outcome is that a player will *never use dominated strategies*. So for practical purpose a dominated strategy never shows up in the game and we can safely discard it and play with a game with a smaller number of strategies. There is no reason why this process cannot sometimes be iterated leading to smaller and smaller games. This principle is called the *iterated elimination of dominated strategies*

Example: Solving a game by iterated elimination of dominated strategies Let us revisit the game given at the beginning of the lecture with payoff matrices

$$P_R = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 5 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \end{array} \qquad P_C = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 4 & 3 \\ 4 & 3 & 2 \end{pmatrix} \end{array}$$

Inspection of the matrices shows that strategy b dominates strategy c for the column player and so we omit c and find the payoff matrices

$$P_R = \begin{array}{c} \\ I \\ II \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix} \end{array} \qquad P_C = \begin{array}{c} \\ I \\ II \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 4 \\ 4 & 3 \end{pmatrix} \end{array}$$

Now strategy I dominates II for the row player and so omitting II we have

$$P_R = I \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 5 & 5 \end{pmatrix} \end{array} \qquad P_C = I \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 4 \end{pmatrix} \end{array}$$

There is no more choice for the row player and we see that the best situation is for R to play I and for C to play b. ∎

Example: Consider the following game which has an arbitrary number of participants, say n of them. Each player announces a number between 1 and 100 and the winner is the one whose number is closest to half the average of these numbers, that is to the number

$$\frac{1}{2}\frac{k_1 + \cdots + k_n}{n}$$

For example for three players playing say, 1, 31, and 100, half the average is 22 and 30 will be the winner.

Before you read on you may want to make an experiment and play this game with your friends. What will they do? If you think about this game for a minute you may argue as follows: if everybody plays without thinking, the average will be around 50 and so I should play around 25. But if everybody makes the same argument the average will be 25 and so I should play around 12.5 and so on... This lead to argue that 1 is the dominant strategy.

This is correct but it is not true that 1 dominates every other strategy, in the previous example 1 is beaten by 31. But one can argue by eliminating iteratively dominated strategies. For example, irrespective of what the players do the average will be no more than 50 and so if you play say more than 50, say 99, then the distance between your number and the half the average is

$$99 - \frac{1}{2}\frac{99 + \cdots + k_n}{n}$$

Elementary algebra show then playing 99 is always better than playing hundred and so we can eliminate 100. You can repeat the argument and show that playing 1 is the rational strategy for this game.

You may want to try and play this game with your friends. Are they playing the dominating strategy? ∎