Week 4: Gambler's ruin and bold play

Random walk and Gambler's ruin. Imagine a walker moving along a line. At every unit of time, he makes a step left or right of exactly one of unit. So we can think that his position is given by an integer $n \in \mathbf{Z}$. We assume the following probabilistic rule for the walker starting at n

$$\begin{aligned} \text{if in position } n \qquad & \text{move to } n+1 \text{ with probability } p \\ & \text{move to } n-1 \text{ with probability } q \\ & \text{stay at } n \text{ with probability } r \end{aligned} \tag{1}$$

with

$$p + q + r = 1$$

Instead a walker on a line you can think of a gambler at a casino making bets of \$1 at certain game (say betting on red on roulette). He start with a fortune of \$n. With probability p he doubles his bet, and the casino pays him \$1 so that he increase his fortune by \$1. With probability q he looses and his fortune decreases by \$1, and with probability r he gets his bet back and his fortune is unchanged.

As we have seen in previous lectures, in many such games the odds of winning are very close to 1 with p typically around .49. Using our second order difference equations we will show that even though the odds are only very slightly in favor of the casino, this enough to ensure that in the long run, the casino will makes lots of money and the gambler not so much. We also investigate what is the better strategy for a gambler, play small amounts of money (be cautious) or play big amounts of money (be bold). We shall see that being bold is the better strategy if odds are not in your favor (i.e. in casino), while if the odds are in your favor the better strategy is to play small amounts of money.

We say that the game is

$$\begin{array}{cl} \text{fair} & \text{if } p = q \\ \text{subfair} & \text{if } p < q \\ \text{superfair} & \text{if } p > q \end{array}$$

The gambler's ruin equation: In order to make the previous problem precise we imagine the following situation.

- You starting fortune if \$j.
- In every game you bet \$1.

- Your decide to play until you either loose it all (i.e., your fortune is 0) or you fortune reaches \$N and you then quit.
- What is the probability to win?

We denote by A_j the event that you win starting with \$j.

$$\begin{aligned} x_j &= P(A_j) = \text{Probability to win starting with } j \\ &= \text{Probability to reach } N \text{ before reaching 0 starting from } j \end{aligned} \tag{2}$$

To compute x_j we use the formula for conditional probability and condition on what happens at the first game, win, lose, or tie. For every game we have

$$P(\text{win}) = p, \quad P(\text{lose}) = q, \quad , P(\text{tie}) = r$$

We have

$$\begin{aligned} x_j &= P(A_j) \\ &= P(A_j|\text{win})P(\text{win}) + P(A_j|\text{lose})P(\text{lose}) + P(A_j|\text{tie})P(\text{tie}) \\ &= \quad x_{j+1} \times p \quad + \quad x_j \times q \quad + \quad x_{j-1} \times r \end{aligned} \tag{3}$$

since if we win the first game, our fortune is then $j+1$, and so $P(A_j|\text{win}) = P(A_j + 1)$ is simply x_{j+1}, and son on....

Note also that we have $x_0 = P(A_0) = 0$ since we have then nothing more to gamble and $x_N = P(A_N)$ since we have reached our goal and then stop playing. Using that $p+q+r=1$ we can rewrite this as the second order equation

Gambler's ruin $\quad px_{j+1} - (p+q)x_j + qx_{j-1} = 0\,, \quad x_0 = 0, x_N = 1$

With $x_j = \alpha^j$ we find the quadratic equation

$$p\alpha^2 - (p+q)\alpha + q = 0$$

with solutions

$$\alpha = \frac{-p \pm \sqrt{(p+q)^2 - 4pq}}{2p} = \frac{-p \pm \sqrt{p^2 + q^2 - 2pq}}{2p} = \frac{-p \pm \sqrt{(p-q)^2}}{2p} = \begin{cases} 1 \\ q/p \end{cases}$$

If $p \neq$ we have two solutions and and so the general solution is given by

$$x_n = C_1 1^n + C_2 \left(\frac{q}{p}\right)^n$$

We will consider the case $p = q$ later. To determine the constants C_1 and C_2 we use that

$$x_0 = 0\,, \qquad \text{and } x_N = 1\,.$$

which follow from the definition of x_j as the probability to win (i.e. reaching a fortune of N) starting with a fortune of j. We find

$$0 = C_1 + C_2\,, \qquad 1 = C_1 + C_2 \left(\frac{q}{p}\right)^N$$

which gives

$$C_1 = -C_2 = \left(1 - \left(\frac{q}{p}\right)^N\right)^{-1}\,,$$

and so we find

$$\boxed{\textbf{Gambler's ruin probabilities} \quad x_n = \frac{1-(q/p)^n}{1-(q/p)^N} \quad p \neq q}$$

To give an idea on how this function look like let us take

$$q = .51,\ p = 0.49,\ r = 0,$$

see figure 1 and let us pick N=100. That is we start with a fortune of j and wish to reach a fortune of 100. We have for example.

$$x_{10} = 0.0091, \quad x_{50} = 0.1191, \quad x_{75} = 0.3560, \quad x_{83} = 0.4973.$$

That is starting with \$10 the probability to win \$100 before losing all you money is only about one in hundred. If you start half-way to your goal, that is with \$50, the probability to win is still a not so great 11 in one hundred and you reach a fifty-fifty chance to win only if you start with \$83.

Bold or cautious? Using the formula for the gambler's ruin we can investigate whether there is a better gambling strategy than betting \$1 repeatedly. For example if we start with \$10 and our goal is to reach \$100 we choose between

- Play in \$1 bets?
- Play in \$10 bets?

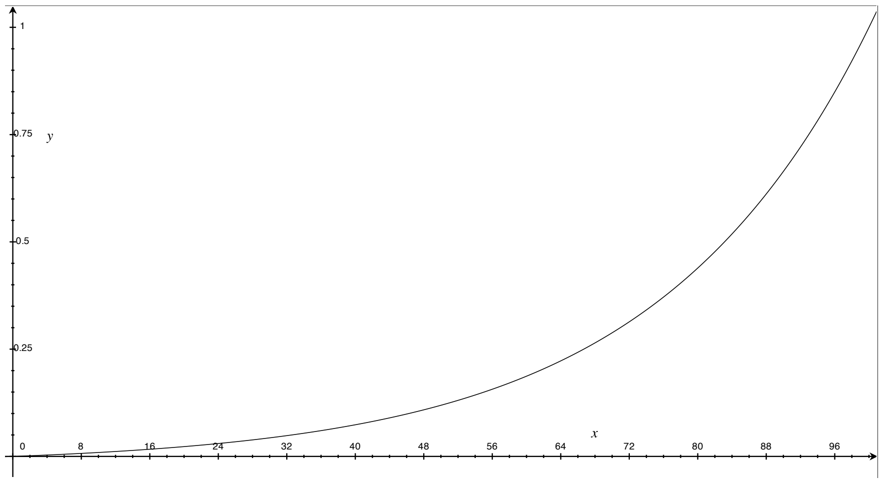

Figure 1: Gambler's ruin probabilities for $n = 100$, $p = 0.49$, $q = 0.51$, $r = 0$

We find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } x_{10} = \frac{1 - (51/49)^{10}}{1 - (51/49)^{100}} = 0.0091$$

while if we bet $10 at each game we use the same formula now with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } x_1 = \frac{1 - (51/49)}{1 - (51/49)^{10}} = 0.0829$$

that is your chance to win is about 8 in hundred, about nine time better than by playing in $1 increments. Based on such arguments it seems clear the best strategy is to be bold if the odds of the game are not in your favor.

If on the contrary the odds are in your favor, even so slightly, say q=.49, and p=.51 then the opposite is true: you should play cautiously. For example with these probabilities and in the same situation as before, starting with $ 10 and with a $100 goal we find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } x_{10} = \frac{1 - (49/51)^{10}}{1 - (49/51)^{100}} = 0.3358$$

while for the other case we use the same formula with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } x_{11} = \frac{1 - (49/51)}{1 - (49/51)^{10}} = 0.1189.$$

In summary we have

> If the odds are in your favor be cautious but if the odds are against you be bold!

Two limiting cases: In order to look at limiting cases we slightly rephrase the problem:

- We start at 0.
- We stop whenever we reach W (W stands for our desired gain) and when we reach $-L$ (L stands for how much money we are willing to lose).

Now $j \in \{-L, -L+1, \cdots, \cdots, W-1, W\}$. We have simply changed variables and so obtain

$$P(-L, W) \equiv P\,(\text{Reach } W \text{ before reaching } -L \text{ starting from } 0) = \frac{1-(q/p)^L}{1-(q/p)^{L+W}}\,.$$

We consider the two limiting cases where W and L go to ∞.

• If L goes to infinity it means that the player is willing to lose an infinite amount of money, that is he has infinite resources and he is trying to reach a gain of W units. If $q/p < 1$ (superfair) then $(q/p)^L \to 0$ as $L \to \infty$ and so $P(-\infty, W) = 1$. On the other hand if $q/p > 1$ (subfair) the ratio is ∞/∞ and we factorize $(q/p)^L$ and find

$$\begin{aligned}
\frac{1-(q/p)^L}{1-(q/p)^{L+W}} &= \frac{(q/p)^L\left((q/p)^{-L}-1\right)}{(q/p)^L\left((q/p)^{-L}-(q/p)^W\right)} \\
&= \frac{(q/p)^{-L}-1}{(q/p)^{-L}-(q/p)^W} \longrightarrow (p/q)^W
\end{aligned}$$

So we find

$$\text{Prob that a gambler with unlimited ressources ever gain } W = \begin{cases} 1 & \text{if } q < p \\ (p/q)^W & \text{if } p < q \end{cases}$$

This is *bad news*: even with infinite resources the probability to ever win a certain given amount in a casino is exponential small!

• If W goes to infinity it means that the player has no win limit and he will be playing either forever or until he looses his original fortune of L. If $q/p > 1$ then the denominator goes to infinity while if $q/p < 1$ it goes to 1. Thus we have

$$\text{Prob that a gambler with no win limit plays for ever} \;=\; \begin{cases} 1-(q/p)^L & \text{if } q<p \\ 0 & \text{if } p<q \end{cases}$$

This is *bad news* again: in a casino the probability to play forever is 0.

The Gambler's ruin for fair games: We briefly discuss the case of fair game $p = q$. In that case the equation for the ruin's probabilities x_j simplify to

$$x_{j+1} - 2x_j + x_{j-1}$$

which gives the quadratic equation

$$\alpha^2 - 2\alpha + 1$$

with only one root $\alpha = 1$. So we have only one solution $x_j = C$. To find a second solution after some head-scratching and we try the solution $x_j = Dj$ and indeed we have

$$D(j+1) - 2Dj + D(j-1) \;=\; 0$$

so that the general solution is

$$x_j \;=\; C_1 + C_2 j\,.$$

With $x_0 = 0$ and $x_N = 1$ we find

$$\textbf{Gambler's ruin probabilities} \;\; x_n \;=\; \frac{n}{N} \qquad \text{if } p = q$$

Bold play strategy The gambler's ruin suggests that if the game is subfair one should bet one's entire fortune. Now if you are trying to reach a certain target there is no reason to bet more than necessary to reach that target, for example if your fortune is \$75 and you want to reach \$200 you will bet \$75 but if you want to reach \$100 you will bet only \$25, keeping some money left to try again if necessary. So we define the *bold play stately* by

Bold play : Bet everything you can but no more than necessary

It turns out we can compute the probability to win using the bold play strategy by using a simple algorithm. First it is useful to normalize since there is no difference between wanting to reach \$100 with bets of \$1 or wanting to reach \$1000 with bets of \$10. So we normalize the target fortune to 1 and imagine we start with a fortune of z with $0 < z < 1$

$Q(z) =$ **probability to reach a fortune of** 1 **starting with a fortune of** f **and playing the bold strategy.**

Example: Let us compute $Q(1/2)$: in this case we bet everything to reach 1 and so $Q(1/2) = p$. For $Q(1/4)$ we first bet everything. If we lose, we lose everything while if we win we now have 1/2. Thus we obtain

$$Q(1/4) \,=\, pQ(1/2) \,=\, p^2.$$

For $Q(3/4)$ we bet 1/4 and we condition on the first bet. We have

$$Q(3/4) \,=\, Q(3/4|W)P(W) + Q(3/4|L)P(L) \,=\, p + Q(1/2)q \,=\, p(1+q)$$

since if we start with 3/4 and win, then we win, and if we lose then we are down to 1/2....
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Formula for bold play probabilities: We derive the basic equations for the bold play strategy. If your fortune z is less than 1/2 then you bet z and ends up with fortune of $2z$ if you win and nothing if you loose. So by conditioning we find

$$Q(z) \,=\, Q(z|W)pP(w) + Q(z|L)P(L) \,=\, pQ(2z) + Q(0)q \,=\, pQ(2z)$$

On the other hand if your fortune z exceeds 1/2 you will bet only $1-z$ to reach 1. By conditioning you find

$$Q(z) \,=\, Q(z|W)pP(w) + Q(z|L)P(L) \,=\, Q(1)p + Q(z-(1-z))q \,=\, p + qQ(2z-1)$$

In summary we have

Bold play conditional probabilities

$$\begin{aligned} Q(z) &= pQ(2z) & \text{if } z \le 1/2 \\ Q(z) &= p + qQ(2z-1) & \text{if } z \ge 1/2 \end{aligned}$$

$$Q(0) = 0\,, \quad Q(1) = 1$$

Example: dyadic rational. We say that z is a dyadic rational if z has the form $z = \frac{j}{2^l}$ for some l. Then $Q(z)$ can be computed recursively. For example we have already computed $Q(1/4)$, $Q(2/4)$ and $Q(3/4)$. We obtain using these values

$$\begin{aligned}
Q(1/8) &= pQ(2/8) \;=\; p^3 \\
Q(3/8) &= pQ(6/8) \;=\; p^2(1+q) \\
Q(5/8) &= p + qQ(2/8) \;=\; p + qp^2 \\
Q(7/8) &= p + qQ(6/8) \;=\; p + qp(1+q) = p(1+q+q^2)
\end{aligned}$$

and clearly we could now commute $Q(1/16), Q(3/16)$, etc...

Example. We can compute $Q(z)$ if z is rational and if we are patient enough. Say we want to compute $Q(1/10)$. We try to find a closed system of equations.

$$\begin{aligned}
Q(1/10) &= pQ(2/10) \\
Q(2/10) &= pQ(4/10) \\
Q(4/10) &= pQ(8/10) \\
Q(8/10) &= p + qQ(6/10) \\
Q(6/10) &= p + qQ(2/10)
\end{aligned}$$

We can have an equation for $Q(2/10)$:

$$Q(2/10) \;=\; p^2Q(8/10) \;=\; p^3 + p^2qQ(6/10) \;=\; p^3(1+q) + p^2q^2Q(2/10)$$

so

$$Q(2/10) \;=\; \frac{p^3(1+q)}{1-p^2q^2}$$

and

$$Q(1/10) \;=\; \frac{p^4(1+q)}{1-p^2q^2}$$

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One can show that

Theorem: (Bold play is optimal) Consider a game where by you double your bet with probability p and lose your bet with probability q. If the game is subfair ($p < 1/2$) then the bold play strategy is optimal in the sense that it gives you the highest probability to reach a certain fortune by a series a bets of varying sizes.

Exercise 1: The *Jacobsthal-Lucas numbers* are given $x_0 = 2$, $x_1 = 1$ and then by the recursion relation

$$x_n = x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Show that these numbers satisfies the recursion relation

$$x_{n+1} = 2x_n - 3(-1)^n \, .$$

Exercise 2: At a certain casino game if you bet \$x you will loose your \$x with probability .505 (so your fortune will decrease by x) and win \$2x with probability \$.495 (so your fortune will increase by x). You walk into the casino with \$25 dollars with the goal to get \$500. Compute the probability for you to succeed if you use the following strategies

1. You make repeated \$5 bets until you either win \$500 or you are wiped out.
2. You make repeated \$25 bets until you either win \$500 or you are wiped out.
3. You play bold strategy.

Exercise 3: Compute the bold strategy probability for $Q(1/7)$, $Q(2/7)$, $Q(3/7)$, $Q(4/7)$, $Q(5/7)$, and $Q(6/7)$.

Exercise 4: Suppose you play a series of fair games with bets of fixed size (say 1).

1. If you start with a fortune of j, what is the probability that you play forever?
2. If you have an infinite amount of money, what is the probability that you ever achieve a gain of M units.

Exercise 5: The following game is proposed to you. You bet repeatedly at certain game where you win 1 with probability p, lose 1 with probability q and you stop upon reaching 0 or N. On top of this at every game, independently of the result of your bet there is the probability 1/10 that you win the jackpot in which case your fortune reaches N immediately and you stop.

1. Argue that the probability to win starting at j satisfies the second order equation

$$x_j = \frac{1}{10} + \frac{9}{10} p x_{j+1} + . \frac{9}{10} q x_{j-1} \, , \qquad q_0 = 0, q_N = 1$$

2. Compute q_j. To get nice formula compute what happens to q_j when $N \to \infty$.