Stochastic Processes Spring 2020: Homework 5

Exercise 1 (An algorithmic tool to compute the stationary distribution) If the state space not small it can be tedious to compute the stationary distribution, i.e., to solve $\pi P = \pi$ by hand. We give an alternative formula for $\pi$ which is very easy to implement on a computer. We let $I$ be the identity matrix and we let $M$ be the matrix whose all entries are $M(i, j) = 1$. Prove

Theorem 1 If $X_n$ is an irreducible Markov chain with transition probabilities matrix $P$. Then the unique stationary distribution $\pi$ is given by

$$\pi = (1, 1, \cdots, 1) (I - P + M)^{-1}.$$ 

To justify this

1. Assume first that the matrix $(I - P + M)$ is invertible. Show that $\pi = (1, 1, \cdots, 1) (I - P + M)^{-1}$ is the stationary distribution

2. You need next to prove that $(I - P + M)$ is invertible. This is equivalent to prove that $(I - P + M)x = 0$ implies $x = 0$. To do this

   (a) Multiply $(I - P + M)x = 0$ by on the left by $\pi$ and deduce from this that $Mx = 0$. Thus $Px = x$.

   (b) Use that the only solutions of $Px = x$ are of the form $x = c(1 \cdots, 1)^T$. Conclude.

Exercise 2 (Card shuffling) Suppose you have a deck of 52 cards. You shuffle the cards by picking a card at random and placing it on top of the deck. This defines a Markov chain.

1. What is the state space?

2. Show that Markov chain defined in this way is irreducible and aperiodic.

3. Find the stationary distribution.

4. Shuffle the deck of cards every second. What is the average time (in years) until the deck returns to the original order?

Exercise 3 On a chessboard compute the expected number of moves it takes a knight starting in of the four corner of the chessboard to return to its initial position if we assume that that at each play it is equally likely to choose any of its legal move.

Exercise 4 For a sequence $\{a_n\}_{n=0}^\infty$ define the sequence

$$b_n = \frac{a_0 + \cdots + a_{n-1}}{n}$$

Show that if $\lim_{n \to \infty} a_n = a$ then $\lim_{n \to \infty} b_n = a$ but that the converse is not true.
Exercise 5 Consider a Markov chain with state \( \{1, \ldots, 6\} \) and transition matrix
\[
P = \begin{pmatrix}
0 & .3 & 0 & 0 & .7 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & .4 & 0 & 0 & .6 & 0 \\
.4 & 0 & .6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

1. Is the chain irreducible?

2. What is the period \( d \) of the chain?

3. Reorder the state and write \( P \) in the canonical form like in the classnotes.

4. Compute \( P^d \) and find all the \( d \) stationary distribution \( \pi_1, \ldots, \pi_d \) of \( P^d \). Find from this the stationary distribution \( \pi \) of \( P \).

5. What is the behavior of the sequence \( P^n(1, j) \) as \( n \) becomes large. You can do this by hand or using a computer if you prefer.

Exercise 6 (Computer exercise) Consider the Markov chain with transition probabilities
\[
P = \begin{pmatrix}
.5 & .4 & 0 & .1 \\
.3 & .3 & .3 & .1 \\
0 & 0 & .4 & .6 \\
.25 & .25 & 0 & .5
\end{pmatrix}.
\]

1. Write down a program which compute the stationary distribution using the formula of Exercise 1.

2. Write down a program which generate a sample of size \( N \) for the Markov chain and return an estimate for the stationary distribution based on that sample. Represent your estimate for \( \pi \) using an histogram and compare with the exact solution obtained in 1.

Exercise 7 (Parameter estimation for a Markov chain)

1. Suppose you observe a sample of a Markov chain \( X_1, X_2, \ldots, X_N \) of length \( N \) but you don’t know anything else about the Markov chain. If you assume that the Markov chain is irreducible build an estimator for \( P(i, j) \) based on the sample \( X_1, X_2, \ldots, X_N \).

   Hint: Use the the result in Homework 3, Exercise 3.

2. Test your estimator using the sample produced in Exercise 6.

Exercise 8 Consider the Ehrenfest urn model in which \( M \) molecules are distributed among two urns and each time point one molecule is chosen at random, removed from its urn, and placed in the other urn. Let \( X_n \) be the number of molecules in urn 1 after the \( n \) switch.

1. Show that \( \pi(k) = \binom{M}{k} \frac{1}{2^M} \) is the unique stationary distribution.

2. Is the chain aperiodic? What is the period.
3. Let $\mu_n = E[X_n]$. Show that

- $\mu_{n+1} = 1 + \left( \frac{M - 2}{M} \right) \mu_n$
- $\mu_n = \frac{M}{2} + \left( \frac{M - 2}{M} \right)^n \left( E[X_0] - \frac{M}{2} \right)$