Stochastic Processes Spring 2020: Homework 2

Exercise 1 Let $X = (X_1, \cdots, X_d)$ be a random vector with independent $X_i \sim N(0, 1)$ and $U \sim \mathcal{U}([0, 1])$. As we have seen in class $Y = \frac{X}{|X|}$ is uniformly distributed on the hypersphere and $Z = U^{1/d} Y$ is uniformly distributed on the ball. Show the following

1. If $X_1, \cdots, X_{d+2} \sim N(0, 1)$ then
   \[
   \sqrt{\frac{\sum_{k=1}^{d} X_i^2}{\sum_{k=1}^{d+2} X_i^2}}
   \]
   has the same distribution as $U^{1/d}$ where $U \sim \mathcal{U}([0, 1])$. Hint: The ratio under the square root is a beta random variable (see exercise 1 and exercise 2).

2. Using part 1, show that if $(Y_1, \cdots, Y_{d+2})$ be uniform distributed on the sphere $\{y_1^2 + \cdots + y_{d+2}^2 = 1\}$ then $(Y_1, \cdots, Y_d)$ is uniformly distributed on the ball $\{y_1^2 + \cdots + y_d^2 \leq 1\}$.

Exercise 2 Show that if $X_i$ are IID random variables with $E[X_i] = \mu$ and $\text{var}[X_i] = \sigma^2$ and $S_N = X_1 + \cdots + X_N$ then the sample variance

\[
V_N^2 = \frac{1}{N-1} \sum_{k=1}^{N} (X_i - \frac{S_N}{N})^2
\]

is an unbiased estimator of $\sigma^2$, that is $E[V_N^2] = \sigma^2$ and $\lim_{N \to \infty} V_N^2 = \sigma^2$ with probability 1.

Exercise 3 Write down a Monte-Carlo program to compute $\pi$ which tracks down both the estimator for $I_N$ and the sample variance $V_N^2$. Use this information to compute $\pi$ with an error of at most $10^{-5}$ and a confidence of % 95 (using the CLT).

Exercise 4 Consider the problem of of computing the average distance between two randomly chosen points on (a) the unit cube (in 3d) and (b) the unit ball (in 3d). The answer to (a) is known as the Robbins constant

\[
\frac{4 + 17\sqrt{2} - 6\sqrt{3} + 21 \log(1 + \sqrt{2}) + 42 \log(2 + \sqrt{3}) - 7\pi}{105} = 0.6617....
\]

(what the hell?).

1. What are the integrals you are trying to compute?
2. Write down a MCMC algorithm for these two problems.
3. Run the corresponding code.

Exercise 5 Suppose $f$ is a function on the interval $[0, 1]$ with $0 \leq f(x) \leq 1$. Here are two ways to estimate $I = \int_0^1 f(x) dx$.

1. Use the following rejection algorithm: Generate 2 random numbers $U_1, U_2 \sim \mathcal{U}([0, 1])$. If $U_2 \leq f(U_1)$ set $X = 1$ and set $X = 0$ otherwise. Then the estimator is
   \[
   I_N = \frac{1}{N} \sum_{k=1}^{N} X_k.
   \]
   where $X_i$ are IID copies of $X$. 

1
2. Let $U_1, U_2, \cdots$ be i.i.d. uniform random variables on $[0, 1]$ and use the estimator

$$
\hat{I}_N = \frac{1}{N} \sum_{k=1}^{N} f(U_k).
$$

Using the variance decide which of the two estimator $I_N$ and $\hat{I}_N$ is the most efficient one.

**Exercise 6** In class note we expressed the structure function $H(X)$ using the concept of minimal paths $H(X) = \max_A \prod_{j \in A} X_j$, where the maximum is taken over all minimal paths $A$. Recall that a minimal path $A$ is a minimal subset of the components $\{1, \cdots, n\}$ whose functioning ensures that the system is working.

As an alternative we can use **minimal cut sets** $B$ which are a minimal subset of components whose failure ensure that the system is not working. Minimal means here that if for any $i \in B$ we turn $X_i = 1$ then the system is functioning. Write down a formula for the structure function using minimal cut sets.

**Exercise 7** Consider the following network

![Network Diagram](image)

Find all minimal paths and minimal cut sets (see previous exercise) and write the corresponding formulas for the structure function. Compute the probability that the system works (assuming all $p_i = p$).

**Exercise 8** Consider the bridge network depicted below. Let $p_i = 0.2$ for all $i = 1, \cdots, 5$.

![Bridge Network Diagram](image)

1. Find all minimal paths and minimal cut sets and write down the structure function accordingly.

2. Compute the probability that the systems works. You can proceed as in class. Alternatively write

$$
P(\text{system works}) = P(\text{at least one minimal path works})
$$

and then use the inclusion-exclusion formula.
3. Write a Monte-Carlo algorithm which reproduce the result obtained in part 1. How many sample should you use?

**Exercise 9** It is natural to assume that a structure function is non-decreasing in every variable $X_i$ (that is turning an component on will not shut the system down...). Show that under this assumption the probability that the system works is non-decreasing in every variable $p_i$ (the probability that component $i$ is working). *Hint: Condition on the value of $X_i$.  
