Final Exam Practice: hints and answers Math 235 Spring 2007

Name:

1. a: Solve the equation using row operations

$$x\begin{pmatrix}1\\0\\-1\end{pmatrix}+y\begin{pmatrix}0\\2\\1\end{pmatrix}+z\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\7\\1\end{pmatrix}.$$

Set up the augmented matrix and use Gaussian elimination:

$$\begin{pmatrix}1&0&-1&:&-2\\0&2&1&:&7\\-1&1&0&:&1\end{pmatrix}\rightarrow\text{Show your work}\rightarrow\begin{pmatrix}1&0&0&:&1\\0&1&0&:&2\\0&0&1&:&3\end{pmatrix}.$$

Thus the solution is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

b: Compute the dot product of the vectors $u=\begin{pmatrix}1\\0\\-3\\-1\end{pmatrix}$, $v=\begin{pmatrix}-1\\-2\\3\\1\end{pmatrix}$ Are these vectors perpendicular? Why?

$$u\cdot v=1(-1)+0(-2)+(-3)3+(-1)(1)=-11.$$

Since the dot product is not 0, the vectors are not perpendicular.

c: Compute the matrix products AB and BA if possible:

$$A=\begin{pmatrix}-1&0&3\\-2&5&3\\0&3&1\end{pmatrix}$$

and

$$B=\begin{pmatrix}1&0&7\\-2&3&4\end{pmatrix}.$$

The product AB does not make sense because the number of columns of A does not equal the number of rows of B. The product BA is

$$BA=\begin{pmatrix}-1&21&10\\-4&27&7\end{pmatrix}.$$

d: For what vectors $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ does the equation $Ax = v$ have a solution if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Set up the augmented matrix and use Gaussian elimination:

$$\begin{pmatrix} 1 & 0 & 1 & : & a \\ 0 & -1 & 2 & : & b \\ 1 & -1 & 3 & : & c \end{pmatrix} \rightarrow \text{Show your work} \rightarrow \begin{pmatrix} 1 & 0 & 1 & : & a \\ 0 & 1 & -2 & : & -b \\ 0 & 0 & 0 & : & c-a-b \end{pmatrix}.$$

It follows that the system has solutions if $c - a - b = 0$.

2. Let A be the matrix

$$\begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}.$$

This represents a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ with respect to the basis $E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Let F be the basis of $\mathbb{R}^3$ given by $\left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. The inverse of the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

is

$$\begin{pmatrix} 2/9 & 1/9 & -2/9 \\ -1/9 & 4/9 & 1/9 \\ 5/9 & -2/9 & 4/9 \end{pmatrix}.$$

a: Let $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. What are the coordinates of w with respect to F.

$$[w]_F = \begin{pmatrix} 2/9 & 1/9 & -2/9 \\ -1/9 & 4/9 & 1/9 \\ 5/9 & -2/9 & 4/9 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 4/9 \\ 7/9 \end{pmatrix}.$$

b: What is the matrix of the linear transformation T with respect to the basis F. Write your answer as a product of matrices, but do not compute the product.

$$[T]_{F\leftarrow F} = [I]_{F\leftarrow E}[T]_{E\leftarrow E}[I]_{E\leftarrow F}$$
$$= \begin{pmatrix} 2/9 & 1/9 & -2/9 \\ -1/9 & 4/9 & 1/9 \\ 5/9 & -2/9 & 4/9 \end{pmatrix} \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

3. a: What does it mean for a basis of $\mathbb{R}^3$ to be orthonormal.

A basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ is orthonormal if $u_i \cdot u_j = 0$ for $1 \leq i \neq j \leq 3$ and $u_i \cdot u_i = 1$ for $1 \leq i \leq 3$. b: Let $F = \{f_1, f_2, f_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let T be reflection across the plane spanned by f_1 and f_2. What is the matrix of T with respect to the basis F?

$$[T]_{F\leftarrow F} = \begin{pmatrix} \vdots & \vdots & \vdots \\ [T(f_1)]_F & [T(f_2)_F] & [T(f_3)]_F \\ \vdots & \vdots & \vdots \end{pmatrix}$$
$$= \begin{pmatrix} \vdots & \vdots & \vdots \\ [f_1]_F & [f_2]_F & [-f_3]_F \\ \vdots & \vdots & \vdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

4. Let P_2 be the vector space of polynomials of degree less than or equal to 2. Which of the following are subspaces of P_2. Explain your reasoning. These explanations are the most important part of the questions.

a: $\{f \in P_2 | f(-1) = 1\}$

This is not a subspace of P_2 because it does not contain the zero polynomial $Z(x) = 0$.

b: $\{f \in P_2 | f(-1) = 0\}$

This is a subspace of P_2. It clearly contains $Z(x) = 0$. If f and g are in this set then

$$(f + g)(-1) = f(-1) + g(-1) = 0 + 0 = 0,$$

hence $f + g$ is in the set. Finally, if f is in the set and c is a scalar,

$$(cf)(-1) = cf(-1) = c \cdot 0 = 0,$$

hence cf is in the set.

c: $\{f \in P_2 | f'' + 2f' = 0\}$ This is a subspace of P_2. It clearly contains $Z(x) = 0$. If f and g are in this set then

$$(f + g)'' + 2(f + g)' = f'' + g'' + 2f' + 2g' = 0 + 0 = 0,$$

hence $f + g$ is in the set. Finally, if f is in the set and c is a scalar,

$$(cf)'' + 2(cf)' = cf'' + 2cf' = c(f'' + 2f') = c \cdot 0 = 0,$$

hence cf is in the set.

5. a: Define what it means for a subset of a vector space to be a basis of that subspace.

   A subset $\mathcal{B}$ of a subspace V is a basis for V if $\mathcal{B}$ is linearly independent and $\text{span}(\mathcal{B}) = V$.

   b: Is $\{1, (t-1), (t-1)^2, (t-1)^3\}$ a basis of P_3. Here P_3 denotes the vector space of all polynomials of degree less than or equal to 3. Why? Again note that explaining why is the important part of the question.

   Since P_3 is 4 dimensional, it suffices to show that $\mathcal{B} = \{1, (t-1), (t-1)^2, (t-1)^3\}$ is linearly independent. Suppose that $\phi(t) = a1 + b(t-1) + c(t-1)^2 + d(t-1)^3$. We need to show that if ϕ is the zero function then $a = b = c = d = 0$. Suppose $\phi(t) = 0$. Then

$$\begin{aligned} \phi(1) &= a = 0 \\ \phi'(1) &= b = 0 \\ \phi''(1) &= 2c = 0 \\ \phi'''(1) &= 6d = 0. \end{aligned}$$

   Thus $a = b = c = d = 0$.

6. a: Let V, W be vector spaces. Define what it means for a function $F : V \to W$ to be a linear transformation.
   F is a linear transformation if

   (a) $F(0_V) = 0_W$, where 0_V is the zero vector in V and 0_W is the zero vector in W.
   (b) $F(f + g) = F(f) + F(g)$ for every $f, g \in V$.
   (c) $F(cf) = cF(f)$ for every $f \in V$ and $c \in \mathbb{R}$.

   b: Are the following linear transformations? Why? Note that the why part of the question is very important.
   b1: $F : P_2 \to P_2, p \mapsto p'' - 3p$
   F is a linear transformation. To see this, we check the definition:

   (a) $F(Z) = Z'' - 3Z = Z$, where Z is the zero polynomial $Z(t) = 0$.
   (b) For every $f, g \in P_2$,

$$\begin{aligned} F(f + g) &= (f + g)'' - 3(f + g) \\ &= f'' + g'' - 3f - 3g \\ &= f'' - 3f + g'' - 3g \\ &= F(f) + F(g). \end{aligned}$$

(c) For every $f \in P_2, c \in \mathbb{R}$,

$$\begin{aligned} F(cf) &= (cf)'' - 3(cf) \\ &= cf'' + 3cf \\ &= c(f'' - 3f) \\ &= cF(f). \end{aligned}$$

b2: $F : P_2 \to \mathbb{R}, p \mapsto p(2)$
F is a linear transformation. To see this, we check the definition:

(a) $F(Z) = Z(2) = 0$, where Z is the zero polynomial $Z(t) = 0$.

(b) For every $f, g \in P_2$,

$$\begin{aligned} F(f+g) &= (f+g)(2) \\ &= f(2) + g(2) \\ &= F(f) + F(g). \end{aligned}$$

(c) For every $f \in P_2, c \in \mathbb{R}$,

$$\begin{aligned} F(cf) &= (cf)(2) \\ &= cf(2) \\ &= cF(f). \end{aligned}$$

b3: $F : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, A \to A^2 - A$. F is not a linear transformation. We can easily check that F does not respect the scalar multiplication. In particular, let I be the 2×2 identity matrix. Then one computes that $F(2I) = 3I \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 2F(I)$.

7. Consider the differential equation $y'' + y = 0$.
a: The functions $y_1(t) = \cos(t), y_2(t) = \sin(t)$ are solutions to this differential equation. Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that have arbitrarily many derivatives. Let $D : V \to V$ be defined by $D : f \mapsto f'' + f$. What properties of D insure that the functions $a_1y_1 + a_2y_2$ for any $a_1, a_2 \in \mathbb{R}$ also solutions to our differential equation?
The fact that D is a linear transformation from $C^\infty \to C^\infty$ guarantees that $a_1y_1 + a_2y_2$ is a solution for any $a_1, a_2 \in \mathbb{R}$.

$$\begin{aligned} D(a_1y_1 + a_2y_2) &= a_1D(y_1) + a_2D(y_2) \quad \text{by the linearity of } D. \\ &= 0 + 0, \quad \text{since } y_1, y_2 \text{ are solutions to the differential equation.} \\ &= 0 \end{aligned}$$

Another way to phrase this is as follows. Since D is a linear transformation, the kernel of D is a subspace. In particular, $\ker(D)$ is closed under taking linear combinations. We note that $\ker(D)$ is precisely the space of solutions to our differential equation. Thus $y_1, y_2 \in \ker(D)$ implies $a_1y_1 + a_2y_2 \in \ker(D)$.

b: The function $y(t) = t + 1$ is a solution to the differential equation $y'' + y = t + 1$ (We are giving you this fact; you do not have to show this is the case). What are all the solutions to the differential equation $y'' + y = t + 1$.
The solutions are

$$\{t + 1 + a_1 y_1 + a_2 y_2 \mid a_1, a_2 \in \mathbb{R}\}.$$

c: Justify your answer to part (b) of this question.

We use the linearity of D. Suppose f is a solution to $y'' + y = t + 1$. Then $f(t) = t + 1 + junk$. For f to be a solutions, we see that

$$\begin{aligned} t + 1 = D(f) &= D(t + 1 + junk) \\ &= D(t + 1) + D(junk) \\ &= t + 1 + D(junk). \end{aligned}$$

Thus for f to be a solution, we need that $junk \in \ker(D)$, which was computed in part a: to be $a_1 y_1 + a_2 y_2 \mid a_1, a_2 \in \mathbb{R}$.

8. a: State the rank-nullity theorem. Define rank and nullity.

Let $T : V \to W$ be a linear transformation between vector spaces V and W. Then

$$\text{rank}(T) + \text{null}(T) = \dim(V).$$

The *rank of* T, denoted $\text{rank}(T)$ is the dimension of the image of T. The *nullity of* T, denoted $\text{null}(T)$ is the dimension of the kernel of T.

b: Show that any differential equation of the form

$$\frac{d^2y}{dx^2} + y = f(x)$$

has a solution for any polynomial $f(x) \in P_2$.

Let D be the linear transformation that sends $g \mapsto g'' + g$. We are being asked to show that $D(y) = f$ has a solution for every $finP_2$. In other words, we need to show that every $f \in P_2$ is in the image of D. View D as a linear transformation from P_2 to P_2. Let $\mathcal{B} = \{1, x, x^2\}$ be a basis for P_2. Then in theses coordinates,

$$[D]_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which clearly has rank 3. In particular, every $f \in P_2$ is in the image of D and hence for every $f \in P_2$, $D(y) = f$ has a solution.

9. Let T be the linear transformation from P_2 to P_2 given by

$$f(x) \mapsto f'' - 2f.$$

Find the matrix of T with respect to the basis $\mathcal{B} = \{1, x-1, (x-1)^2\}$.

$$\begin{aligned} [T]_{\mathcal{B}\to\mathcal{B}} &= \begin{pmatrix} \vdots & \vdots & \vdots \\ [T(1)]_{\mathcal{B}} & [T(x-1)]_{\mathcal{B}} & [T((x-1)^2)]_{\mathcal{B}} \\ \vdots & \vdots & \vdots \end{pmatrix} \\ &= \begin{pmatrix} \vdots & \vdots & \vdots \\ [-2]_{\mathcal{B}} & [-2(x-1)]_{\mathcal{B}} & [2-2(x-1)^2)]_{\mathcal{B}} \\ \vdots & \vdots & \vdots \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

10. Define an inner product (or equivalently, a dot product) on P_2 by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Find an orthonormal basis of P_2 with respect to this inner product.

We fist find a basis of P_2, then use Gram-Schmidt to create an orthonormal basis. Fix a basis $\mathcal{B} = \{f_1(x) = 1, f_2(x) = x - \frac{1}{2}, f_3(x) = (x - \frac{1}{2})^2\}$. Recall that the Gram-Schmidt process yields an orthonormal basis $\{u_1, u_2, u_3\}$, where

$$\begin{aligned} u_1 &= \frac{1}{\|f_1\|} f_1 \\ u_2 &= \frac{1}{\|f_2^\perp\|} f_2^\perp, \quad \text{where } f_2^\perp = f_2 - \langle f_2, u_1\rangle u_1 \\ u_3 &= \frac{1}{\|f_3^\perp\|} f_3^\perp, \quad \text{where } f_3^\perp = f_3 - \langle f_3, u_1\rangle u_1 - \langle f_3, u_2\rangle u_2. \end{aligned}$$

We use the definitions to compute the relevant inner products:

$$\|f_1\|^2 = \langle f_1, f_1\rangle = \int_0^1 1dx = 1,$$

and so $u_1(x) = 1$. To compute u_2,

$$\begin{aligned} \langle f_2, u_1\rangle &= \int_0^1 x - \frac{1}{2}dx = 0, \quad \text{hence} \\ f_2^\perp(x) &= x - \frac{1}{2} \quad \text{and} \\ \|f_2^\perp\|^2 &= \langle f_2^\perp, f_2^\perp\rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}. \end{aligned}$$

It follows that

$$u_2(x) = 2\sqrt{3}(x - \frac{1}{2}).$$

To compute u_3,

$$\begin{aligned}
\langle f_3, u_1\rangle &= \int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12} \quad \text{and} \\
\langle f_3, u_2\rangle &= 2\sqrt{3}\int_0^1 (x-\frac{1}{2})^3 dx = 0, \quad \text{hence} \\
f_3^{\perp}(x) &= (x-\frac{1}{2})^2 - \frac{1}{12} \quad \text{and} \\
\|f_3^{\perp}\|^2 = \langle f_3^{\perp}, f_3^{\perp}\rangle &= \int_0^1 ((x-\frac{1}{2})^2 - \frac{1}{12})^2 dx = \frac{7}{20}.
\end{aligned}$$

It follows that

$$u_3(x) = \frac{2\sqrt{5}}{\sqrt{7}}\left((x-\frac{1}{2})^2 - \frac{1}{12}\right).$$

11. The two vectors $u_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ are orthogonal.

a: Verify this fact.
To verify this fact, we compute the dot product to see that it is 0.

$$u_1 \cdot u_2 = 1(1) + (-1)(3) + (2)(1) = 0.$$

b: Find the vector in the space spanned by u_1 and u_2 that is closest to the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Let $v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $V \subset \mathbb{R}^3$ the space spanned by u_1 and u_2. Then vector in V that is closest to v is the projection of v onto the V. Since u_1 and u_2 are orthogonal, we can get an orthonormal basis of V by dividing by the respective lengths. In particular,

$$\left\{ \tilde{u}_1 \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \tilde{u}_2 = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$$

is an orthonormal basis of V.

We compute the projection of v onto V is then

$$(v \cdot \tilde{u}_1)\tilde{u}_1 + (v \cdot \tilde{u}_2)\tilde{u}_2 = \begin{pmatrix} \frac{1}{6} - \frac{8}{11} \\ -\frac{1}{6} + \frac{24}{11} \\ \frac{1}{3} + \frac{8}{11} \end{pmatrix} = \begin{pmatrix} -\frac{37}{66} \\ \frac{133}{66} \\ \frac{35}{33} \end{pmatrix}.$$

12. Using the method of expansion by minors compute the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Note that you are told to use expansion by minors, and so you must use that method to receive credit.

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} &= 1 \det \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} - 0 \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix} \\ &= -4 + 0 + 3(2) \\ &= 2. \end{aligned}$$

13. a: Define eigenvalue and eigenvector of a matrix.

Let A be a $n \times n$ matrix. Then a non-zero vector $v \in \mathbb{R}^n$ is an *eigenvector of A* with associated *eigenvalue* λ if

$$Av = \lambda v.$$

b: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 7/5 & -1/5 \\ -6/5 & 8/5 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic polynomial

$$\begin{aligned} f_A(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} 7/5 - \lambda & -1/5 \\ -6/5 & 8/5 - \lambda \end{pmatrix} \\ &= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2). \end{aligned}$$

It follows that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. An eigenvector v_1 associated to $\lambda_1 = 1$ is a non-zero vector in $\ker(A - I) = \ker \begin{pmatrix} 2/5 & -1/5 \\ -6/5 & 3/5 \end{pmatrix}$. By inspection, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector associated to eigenvalue 1. Similarly, an eigenvector v_2

associated to $\lambda_2 = 2$ is a non-zero vector in $\ker(A - 2I) = \ker \begin{pmatrix} -3/5 & -1/5 \\ -6/5 & -2/5 \end{pmatrix}$. By inspection, $v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is an eigenvector associated to eigenvalue 2.

c: Find a matrix B so that $C = BAB^{-1}$ is diagonal. What is the matrix C?

Partial Answer: the eigenvalues of the matrix A are $1, 2$, corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Note that A is diagonalizable, and an eigenbasis, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$ are the coordinates in which A looks diagonal. In particular,

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = S^{-1}AS, \quad \text{where } S = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$

It follows that the matrix $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = S^{-1} = \begin{pmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{pmatrix}$.

14. Let $C(n)$ denote the coyote population of an imaginary ecosystem after n time intervals have passed. Similarly let $R(n)$ denote the population of this ecosystem after n time intervals. Let

$$A = \begin{pmatrix} 10/21 & 16/21 \\ -8/21 & 46/21 \end{pmatrix}.$$

Assume that

$$\begin{pmatrix} C(n) \\ R(n) \end{pmatrix} = A \begin{pmatrix} C(n+1) \\ R(n+1) \end{pmatrix}.$$

Describe how this ecosystem behaves for different positive initial values $\begin{pmatrix} R(0) \\ C(0) \end{pmatrix}$. In particular indicate for what initial values both species survive and for what initial values only one species survives in the long run.

Partial answer: The eigenvalues of the matrix A are $2/3, 2$. The corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Suppose the initial population is $\begin{pmatrix} C_0 \\ R_0 \end{pmatrix}$. We express this in terms of the eigenvectors as

$$\begin{pmatrix} C_0 \\ R_0 \end{pmatrix} = a \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} C(t) \\ R(t) \end{pmatrix} &= A^t \begin{pmatrix} C_0 \\ R_0 \end{pmatrix} \\ &= a \left(\frac{2}{3}\right)^t \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b(2^t) \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

As t gets large, $\left(\frac{2}{3}\right)^t$ gets small and 2^t gets large, so we see that in order for the populations to both survive, the initial population $\begin{pmatrix} C_0 \\ R_0 \end{pmatrix} = a \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4a + 2b \\ a + b \end{pmatrix}$ must be such that $b > 0$. This is precisely when $C_0 < 4R_0$.