

1 Quick and Dirty Differential Equations

1.1 Two solutions

Let $A, B, C \in \mathbb{R}$. We show how to find two solutions to the differential equation

$$A \frac{d^2 f}{dx^2} + B \frac{df}{dx} + Cf = 0.$$

Guess at solution of the form $f(x) = e^{ax}$. Substitute this into the equation above. We get

$$(Aa^2 + Ba + C)e^{ax} = 0.$$

The above expression can only be the zero function if a is a root of the quadratic equation $Aa^2 + Ba + C = 0$.

If the two roots r_1, r_2 of our quadratic equation are real and distinct, then we get the two solutions to the differential equation $f_1(x) = e^{r_1 x}, f_2(x) = e^{r_2 x}$. If $r_1 = r_2 = r$, then $f_1(x) = e^{rx}, f_2(x) = xe^{rx}$.

If the two roots to the quadratic equation are pure imaginary $\pm is, s \in \mathbb{R}$, then we get the two solutions $f_1(x) = \cos(sx), f_2(x) = \sin(sx)$.

In general, a quadratic with real coefficients has conjugate roots $r \pm is$ and we get the two solutions $f_1(x) = e^{rx} \cos(sx), f_2(x) = e^{rx} \sin(sx)$.

1.2 Linear Operator

Let V be the space of smooth functions on the real line. Define a map or “operator”

$$L : V \longrightarrow V$$

$$L : f \mapsto Af'' + Bf' + Cf.$$

Here $A, B, C \in \mathbb{R}$. Another way we can write this is

$$L = AD^2 + BD + C = (D - r_1)(D - r_2)$$

where $D = \frac{d}{dx}$ is the derivative operator and r_1, r_2 are the roots found above.

Observe that L is linear. This implies that the kernel of L is a subspace. Hence if f_1, f_2 are elements in the kernel of L , that is, solutions to $L(f) = 0$, that is, solutions to the differential equation $A \frac{d^2 f}{dx^2} + B \frac{df}{dx} + Cf = 0$, then the functions $f = af_1 + bf_2$ for arbitrary $a, b \in \mathbb{R}$ are solutions. Thus from two solutions we get a two parameter family of solutions to the differential equation.

1.3 All Solutions

Keep the notation for L, V from above. It is a fact that a solution to $Lf = 0$ is determined by the value of f at $x = 0$ together with the value of f' at $x = 0$. This means that if f, g are both solutions to the differential equation $Lf = 0$ and they have the same value and the same derivative at $x = 0$, then they are the same function.

Suppose we have two solutions f_1, f_2 of $Lf = 0$. Suppose further that we can solve the equations

$$(af_1 + bf_2)(0) = \alpha \tag{1}$$

$$(af_1 + bf_2)'(0) = \beta. \tag{2}$$

Let g be an arbitrary solution to the equation $L(f) = 0$. Let $g(0) = \alpha, g'(0) = \beta$. Then we can find a linear combination of f_1, f_2 so that the linear combination is equal to g at 0 and the derivative of this linear combination at zero is equal to $g'(0)$. Hence the linear combination is equal to g . We have shown that an arbitrary solution is a linear combination of f_1, f_2 .

1.4 One Solution to an Inhomogeneous Equation

Let P_n denote the vector space of polynomials of degree less than or equal to n . We work with $L : V \rightarrow V, f \mapsto Af'' + Bf' + Cf$.

Problem: How do you find a solution to an equation of the form $L(f) = g$, with $g \in P_n$.

We call this an inhomogeneous equation if $g \neq 0$. We call the equation $Lf = 0$ the associated homogeneous equation.

We solve this problem in a particular case. Assume that $L(f) = f'' + 3f$ and that $g \in P_2, g = 2t^2 + 3t - 4$. We look at the linear operator $L_2 : P_2 \rightarrow P_2, f \mapsto f'' + 3f$. We choose a basis for P_2 , namely $\{1, t, t^2\}$. With respect to this basis L_2 can be written as the matrix

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

In terms of our chosen basis, the vector $g \in P_2$ corresponds to the column vector $\begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$. In matrix terms we need to solve

$$A\vec{v} = \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}.$$

The solution to this matrix equation is $\begin{pmatrix} -8/3 \\ 1 \\ 2/3 \end{pmatrix}$. this corresponds to the polynomial $h(t) = -16/9 + t + (2/3)t^2$.

1.5 All Solutions to an Inhomogeneous Equation

We have found one solution to the equation $f'' + f = g$. How to we find the rest of the solutions?

Let f be a solution to $L(f) = 0$. Notice that

$$L(f + h) = L(f) + L(h) = 0 + h = h.$$

Hence we can get more solutions to the inhomogeneous equation by adding solutions to the homogeneous equation to the single solution of the inhomogeneous equation.

Indeed all solutions to the inhomogeneous equation arise this way. Let f be an arbitrary solution to $L(f) = g$. Then

$$L(f - h) = L(f) - L(h) = g - g = 0.$$

Hence $f - h = f_1$ is a solution to the homogeneous equation and $f = h + f_1$.