Review guide for the final exam in Math 233

1 Basic material.

This review includes the remainder of the material for math 233. The final exam will be a cumulative exam with many of the problems coming from the material covered beginning approximately with chapter 15.4 of the book.

We first recall polar coordinates formulas given in chapter 10.3. The coordinates of a point $(x, y) \in \mathbb{R}^3$ can be described by the equations:

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta), \tag{1}$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the origin and $(\frac{x}{r}, \frac{y}{r})$ is $(\cos(\theta), \sin(\theta))$ on the unit circle. Note that $r \geq 0$ and θ can be taken to lie in the interval $[0, 2\pi)$.

To find r and θ when x and y are known, we use the equations:

$$r^2 = x^2 + y^2 \qquad\qquad \tan(\theta) = \frac{y}{x}. \tag{2}$$

Example 1 Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

Solution : Since $r = 2$ and $\theta = \frac{\pi}{3}$, Equations 1 give

$$x = r\cos(\theta) = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin(\theta) = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}.$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 2 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution : If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan(\theta) = \frac{y}{x} = -1.$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$. Thus, one possible answer is $(\sqrt{2}, -\frac{\pi}{4})$; another is $(r, \theta) = (\sqrt{2}, \frac{7\pi}{4})$.

The next theorem describes how to calculate the integral of a function $f(x, y)$ over a polar rectangle. Note that $dA = r\,dr\,d\theta$.

Theorem 3 (Change to Polar Coordinates in a Double Integral) If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\int_R \int f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))r\ dr\ d\theta.$$

Example 4 Evaluate $\int\int_R (3x+4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2 = y^2 = 1$ and $x^2+y^2=4$.

Solution : The region R can be described as

$$R = \{(x,y) \mid y \geq 0,\ 1 \leq x^2+y^2 \leq 4\}.$$

It is a half-ring and in polar coordinates it is given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Therefore, by Theorem 3,

$$\int_R \int (3x+4y^2)dA = \int_0^\pi \int_1^2 (3r\cos(\theta) + 4r^2\sin^2(\theta))r\ dr\ d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos(\theta) + 4r^3\sin^2(\theta))\ dr\ d\theta$$

$$= \int_0^\pi [r^3\cos(\theta) + r^4 sin^2(\theta)]_{r=1}^{r=2}\ d\theta = \int_0^\pi (7\cos(\theta) + 15\sin^2(\theta))\ d\theta$$

$$= \int_0^\pi [7\cos(\theta) + \frac{15}{2}(1-\cos(2\theta))]\ d\theta$$

$$= 7\sin(\theta) + \frac{15\theta}{2} - \frac{15}{4}\sin(2\theta)\Big]_0^\pi = \frac{15\pi}{2}.$$

Example 5 Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z = 1-x^2-y^2$.

Solution : If we put $z=0$ in the equation of the paraboloid, we get $x^2+y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2+y^2=1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2+y^2 \leq 1$. In polar coordinates D is given by $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$. Since $1-x^2-y^2 = 1-r^2$, the volume is

$$V = \int_D \int (1-x^2-y^2)dA = \int_0^{2\pi} \int_0^1 (1-r^2)r\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r-r^3)\ dr\ d\theta = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

The next theorem extends our previous application of Fubini's theorem for type II regions.

Theorem 6 If f continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\int_D \int f(x, y) dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r\ dr\ d\theta$$

The next definition describes the notion of a vector field. We have already seen an example of a vector field associated to a function $f(x, y)$ defined on a domain $D \subset \mathbb{R}^2$, namely the gradient vector field $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. In nature and in physics, we have the familiar examples of the velocity vector field in weather and force vector fields that arise in gravitational fields, electric and magnetic fields.

Definition 7 Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Definition 8 Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Note that a vector field $\mathbf{F}$ on $\mathbb{R}^3$ can be expressed by its component functions. So if $\mathbf{F} = (P, Q, R)$, then:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

We now describe our first kind of "line integral". These type of integrals arise form integrating a function along a curve C in the plane or in $\mathbb{R}^3$. The type of line integral described in the next definition is called a **line integral with respect to arc length**.

Definition 9 Let C be a smooth curve in $\mathbb{R}^2$. Given n, consider n equal subdivisions of lengths Δs_i; let (x_i^*, y_i^*) denote the midpoints of the i-th subdivision. If f is a real valued function defined on C, then the **line integral of** f **along** C is

$$\int_C f(x, y)\ ds = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_i^*, y_i^*)\Delta s_i,$$

if this limit exists.

The following formula can be used to evaluate this type of line integral.

Theorem 10 Suppose $f(x, y)$ is a continuous function on a differentiable curve $C(t)$, $C\colon [a, b] \to \mathbb{R}^2$. Then

$$\int_C f(x, y)\ ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

In the above formula,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

is the speed of $C(t)$ at time t.

Example 11 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution : We can choose x as the time parameter and the equations for C become

$$x = x \qquad y = x^2 \qquad 0 \leq x \leq 1$$

Therefore,

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2x\sqrt{1+4x^2}\,dx = \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{\frac{3}{2}}\Big]_0^1 = \frac{5\sqrt{5}-1}{6}.$$

Actually for what we will studying next, another type of line integral will be important. These line integrals are called **line integrals of f along C with respect to x and y.** They are defined respectively for x and y by the following limits:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta y_i.$$

The following formulas show how to calculate these new type line integrals. Note that these integrals depend on the orientation of the curve C, i.e., the initial and terminal points.

Theorem 12

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt.$$

Example 13 Evaluate $\int_C y^2\,dx + x\,dy$, where $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

Solution : A parametric representation for the line segment is

$$x = 5t-5, \quad y = 5t-3, \quad 0 \leq t \leq 1$$

Then $dx = 5\,dt$, $dy = 5\,dt$, and Theorem 12 gives

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)^2(5dt) + (5t-5)(5\,dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\,dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}.$$

Example 14 Evaluate $\int_C y^2\,dx + x\,dy$, where $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Solution : Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \quad y = y, \qquad -3 \le y \le 2.$$

Then $dx = -2y\,dy$ and by Theorem 12 we have

$$\int_{C_2} y^2 dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^{2} = 40\frac{5}{6}.$$

One can also define in a similar manner the line integral with respect to arc length of a function f along a curve C in $\mathbb{R}^3$.

Theorem 15

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz,$$

where $f(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

The next example demonstrates how to calculate a line integral of a function with respect to x, y and z.

Example 16 Evaluate $\int_c y\,dx + z\ \ dy + x\,dz,$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$ followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

Solution : We write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \le t \le 1.$$

Thus

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)4\,dt + (2 + t)5\,dt$$

$$= \int_0^1 (10 + 29t)\,dt = 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5.$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5 - 5t\rangle$$

or

$$x = 3 \quad y = 4 \quad z = 5 - 5t \quad dz = -5\,dt, \quad 0 \le t \le 1.$$

Then $dx = 0 = dy$, so

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 2(-5)\,dt = -15.$$

Adding the values of these integrals, we obtain

$$\int_{C=C_1\cup C_2} y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5.$$

We now get to our final type of line integral which can be considered to be **a line integral of a vector field**. This type of integral is used to calculate the work W done by a force field F in moving a particle along a smooth curve C.

Theorem 17 If C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on the interval $[a, b]$, then the work W can be calculated by

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt,$$

where $\cdot$ is the dot product.

In general, we make the following definition which is related to the formula in the above theorem.

Definition 18 Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds;$$

here, T is the unit tangent vector field to the parameterized curve C.

Example 19 Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$.

Solution : Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\mathbf{i} - \cos t \sin t\mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}.$$

Therefore, the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^{\frac{\pi}{2}} (-2\cos^2 t \sin t)dt$$

$$= 2\frac{\cos^3 t}{3}\Big]_0^{\frac{\pi}{2}} = -\frac{2}{3}.$$

Example 20 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1.$$

Solution : We have

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$
$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$
$$\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}.$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

$$= \int_0^1 (t^3 + 5t^6)dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big]_0^1 = \frac{27}{28}.$$

Theorem 21 If C in $\mathbb{R}^3$ is parameterized by $\mathbf{r}(t)$ and $F = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz.$$

We now apply the material covered so far on line integrals to obtain several versions of the fundamental theorem of calculus in the multivariable setting. Recall that the fundamental theorem calculus can be written as

$$\int_a^b \mathbf{F}'(x)dx = F(b) - F(a),$$

when $\mathbf{F}'(x)$ is continuous on $[a, b]$.

Theorem 22 Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

For further discussion, we make the following definitions.

Definition 23 A curve $r\colon [a,b] \to \mathbb{R}^3$ (or $\mathbb{R}^3$) **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.

Definition 24 A domain $D \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is **open** if for any point p in D, a small ball (or disk) centered at p in $\mathbb{R}^3$ (in $\mathbb{R}^2$) is contained in D.

Definition 25 A domain $D \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is **connected** if any two points in D can be joined by a path contained inside D.

Definition 26 A curve $\mathbf{r}\colon [a,b] \to \mathbb{R}^3$ (or $\mathbb{R}^2$) is a **simple curve** if it doesn't intersect itself anywhere between its end points ($\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$).

Definition 27 An open, connected region $D \subset \mathbb{R}^2$ is a **simply-connected** region if any simple closed curve in D encloses only points that are in D.

Definition 28 A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function $f(x,y)$; the function $f(x,y)$ is called a **potential function** for $\mathbf{F}$. For example, for $f(x,y) = xy+y^2$, $\nabla f = \langle y, x+2y\rangle$ and so, $\mathbf{F}(x,y) = y\mathbf{i} + (x+2y)\mathbf{j}$ is a conservative vector field.

Definition 29 If F is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D with the same initial and the same terminal points.

We now states several theorems that you should know for the final exam.

Theorem 30 $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem 31 Suppose F is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem 32 If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 33 Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D.$$

Then $\mathbf{F}$ is conservative.

Example 34 Determine whether or not the vector field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$ is conservative.

Solution : Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1.$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\mathbf{F}$ is not conservative by Theorem 32.

Example 35 Determine whether or not the vector field $\mathbf{F}(x, y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.

Solution : Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore, we can apply Theorem 33 and conclude that $\mathbf{F}$ is conservative.

Attention! You will likely have a problem on the final exam which is similar to the one described in the next example.

Example 36 (a) If $\mathbf{F}(x, y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = e^t \sin t\, \mathbf{i} + e^t \cos t\, \mathbf{j}$, $0 \leq t \leq \pi$.

Solution :

(a) From Example 35 we know that $\mathbf{F}$ is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy \tag{3}$$

$$f_y(x, y) = x^2 - 3y^2 \tag{4}$$

Integrating (3) with respect to x, we obtain

$$f(x, y) = 3x + x^2 y + g(y). \tag{5}$$

Notice that the constant of integration is a constant with respect to x, that is, it is a function of y, which we have called $g(y)$.

Next we differentiate both sides of (5) with respect to y:

$$f_y(x, y) = x^2 + g'(y). \tag{6}$$

Comparing (4) and (6), we see that

$$g'(y) = -3y^2.$$

Integrating with respect to y, we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (5), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

(b) To apply Theorem 22 all we have to know are the initial and terminal points of C, namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^{\pi})$. In the expression for $f(x, y)$ in part (a), any value of the constant K will do, so let's choose $K = 0$. Then we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= f(0, -e^{\pi}) - f(0, 1) \\ &= e^{3\pi} - (-1) = e^{3\pi} + 1.\end{aligned}$$

This method is much shorter than the straightforward method for evaluating line integrals described in Theorem 12.

Definition 37 A simple closed parameterized curve C in $\mathbb{R}^2$ always bounds a bounded simply-connected domain D. We say that C is **positively oriented** if for the parametrization $\mathbf{r}(t)$ of C, the region D is always on the left as $\mathbf{r}(t)$ traverses C. Note that this parametrization is the counterclockwise one on the boundary of unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

The next theorem is a version of the fundamental theorem of calculus, since it allows one to carry out a two-dimensional integral on a domain D by calculating a related integral one-dimensional on the boundary of D. There will be at least one final exam problem related to the following theorem.

Theorem 38 (Green's Theorem) Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \int_D\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

An immediate consequence of Green's Theorem are the area formulas described in the next theorem.

Theorem 39 Let D be a simply-connected domain in the plane with simple closed oriented boundary curve C. Let A be the area of D. Then:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx. \tag{7}$$

Example 40 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leq t \leq 2\pi$. Using the third formula in Equation 7, we have

$$\begin{aligned} A &= \frac{1}{2}\int_C x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab. \end{aligned}$$

Example 41 Use Green's Theorem to evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle
$x^2 + y^2 = 9$.

Solution : The region D bounded by C is the disk $x^2+y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\begin{aligned} &\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy \\ &= \int_D\int \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA \\ &= \int_0^{2\pi}\int_0^3 (7-3)\,r\,dr\,d\theta \\ &= 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi. \end{aligned}$$