Review guide for midterm 2 in Math 233

Midterm 2 covers material that begins approximately with the definition of partial derivatives in Chapter 14.3 and ends approximately with methods for calculating the double integral of a function $f(x, y)$ over a domain D described in the xy-plane. See the updated course web page for the exact material covered on this exam.

Definition 1 (Partial Derivatives) If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

We have the following rule for calculating partial derivatives.

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 2 Calculate f_x, f_y for $f(x, y) = x^2 e^{xy} + y^2$.

Solution : We apply the sum, product and chain rules for derivatives, to get:

$$f_x(x, y) = 2xe^{xy} + x^2 e^{xy} y = 2xe^{xy} + x^2 y e^{xy}$$

$$f_y(x, y) = x^2 e^{xy} x + 2y = x^3 e^{xy} + 2y.$$

Definition 3 (Second Partial Derivatives) For $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example 4 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2.$$

Solution : Note:

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y.$$

Therefore,

$$f_{xx} = 6x + 2y^3 \qquad f_{xy} = 6xy^2$$
$$f_{yx} = 6xy^2 \qquad f_{yy} = 6x^2y - 4.$$

Note that in the above example $f_{xy} = f_{yx}$. This is no coincidence and follows from the next theorem that states that under weak conditions on $f(x, y)$, taking partial derivatives is a commutative process.

Theorem 5 (Clairaut's Theorem) Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

The next definition of tangent plane generalizes in a natural way the following equation of the tangent line of a function of 1 variable:

$$y - y_0 = f'(x_0)(x - x_0).$$

Definition 6 (Tangent Plane) Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 7 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution : Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2.$$

Then Definition 6 gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3.$$

The next definition of linear approximation generalizes the linear approximation $L(x)$ of a function $f(x)$ of 1 variable at a point $x_0 = a$:

$$L(x) = f(x) + f'(x)(x - a).$$

Definition 8 (Linear Approximation) The linear approximation of $f(x,y)$ at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example 9 Use the linear approximation $L(x,y)$ to $f(x,y) = xe^{xy}$ at $(1,2)$ to estimate $f(1.1, 1.8)$.

Solution :

$$\begin{aligned} f1,2) &= e^2 \\ f_x(x,y) &= e^{xy} + xye^{xy} \qquad f_y(x,y) = x^2e^{xy} \\ f_x(1,2) &= e^2 + 2e^2 \qquad f_y(1,2) = e^2. \\ L(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2). \end{aligned}$$

Hence,

$$L(1.1, 1.8) = e^2 + (e^2 + 2e^2)(.1) + e^2(-.2).$$

Definition 10 If $z = f(x,y)$, then f is **differentiable** at (a,b) if the change Δz of z can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

The next theorem gives a simple condition for $f(x,y)$ to satisfy in order to be differentiable.

Theorem 11 If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Definition 12 (Total Differential) For $z = f(x,y)$,

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Example 13 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm. in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution : The volume V of a cone with base radius r and height h is $V = \pi r^2 \frac{h}{3}$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = \frac{2\pi r h}{3}\,dr + \frac{\pi r^2}{3}\,dh.$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leq 0.1$, $|\Delta h| \leq 0.1$. To find the largest error in the volume we use the largest error in the measurement of r and of h. Therefore, we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives the estimate

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = \frac{60\pi}{3} = 20\pi.$$

Thus, the maximum error in the calculated volume is about 20π cm$^3 \approx 63$ cm^3.

Example 14 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution : If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz.$$

We are given that $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$. To find the largest error in the volume, we use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980.$$

Thus, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

Theorem 15 (Chain Rule Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Theorem 16 (Chain Rule Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Theorem 17 (The Chain Rule (General Version)) Suppose that u is a differentiable function of n variables $x_1,\ x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1,\ t_2, \ldots, t_m$. Then u is a function of $t_1,\ t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1,\ 2, \ldots, m$.

Example 18 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Solution : The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t).$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$ we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore,

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6.$$

Example 19 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution : If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since $P = 8.31\frac{T}{V}$, with $\frac{\partial P}{\partial T} = \frac{8.31}{V}$ and $\frac{\partial T}{\partial V} = -\frac{8.31T}{V^2}$, then Case 1 of the Chain Rule gives

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial T}\frac{dT}{dt} - \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt} \\ &= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155.\end{aligned}$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

Example 20 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution : Applying Case 2 of the Chain Rule, we get

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t), \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2sre^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t).\end{aligned}$$

Example 21 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin yt$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

Solution : We have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2zy^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t).$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192.$$

Theorem 22 (Implicit Differentiation) Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $F(x, y, z) = 0$, i.e., $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of $f(x, y)$. Then:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Example 23 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution : Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Theorem 22, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

Definition 24 (Directional Derivative) The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of a **unit** vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Definition 25 (Directional Derivative) The **directional derivative** of $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Definition 26 (Gradient) If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Definition 27 (Gradient) For $f(x, y, z)$, a function of three variables,

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The next two theorems give simple rules for calculating the directional derivative of a function in 2 or 3 variables in terms of the gradient of the function.

Theorem 28 If f is a differentiable function of x and y, then f has a directional derivative in the direction of any **unit** vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Theorem 29 If f is a differentiable function of x, y, and z, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b, c \rangle$ and

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

By the above two theorems, we have for any unit vector $\mathbf{u}$,

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||u| \cos(\theta) = |\nabla f| \cos(\theta).$$

Thus, the next theorem holds.

Theorem 30 Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Example 31 Find the directional derivative of the function $f(x, y) = x^2 y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Solution : We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3 \mathbf{i} + (3x^2 y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}.$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}.$$

Therefore, by Theorem 28, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot (\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j})$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}.$$

Theorem 32 Suppose S is a surface determined as $F(x, y, z) = k$ for $k =$ constant. Then ∇F is everywhere normal or orthogonal to S. In particular, if $P = (x_0, y_0, z_0) \in S$, then the equation of the tangent plane to S at p is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (1)$$

Example 33 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Solution : The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

Therefore, we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\frac{2}{3}.$$

Then Equation 1 in Theorem 32 gives the equation of the tangent plane at $(-2, 2, -3)$ as

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0,$$

which simplifies to $3x - 6y + 2z = 18 = 0$.

Since $\nabla F(-2, 1, -3) = \langle -1, 2, -\frac{2}{3}\rangle$, the vector equation of the normal line is:

$$L(t) = \langle -2, 1, -3\rangle + t\langle -1, 2, -\frac{2}{3}\rangle.$$

Definition 34 A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). (This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).) The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \leq f(a, b)$ for all $f(x, y)$ in the domain of f, then f has an **absolute maximum** at (a, b). If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is a **local minimum value**. If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum** at (a, b).

The next theorem explains how to find local maxima and local minima for a function in two variables.

Theorem 35 If f has a local maximum of minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition 36 A point (a, b) is called a **critical** point of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$.

The next theorem gives a method for testing critical points of a function $f(x, y)$ to see if they represent local minima, local maxima or saddle points (a critical point (a, b) is a **saddle** point if the Hessian D defined in the next theorem is negative).

Theorem 37 (Second Derivative Test) Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f). Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy(a,b)}]^2.$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(c) If $D < 0$, then $f(a,b)$ is a saddle point.

To remember the formula for D it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 38 Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution : We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x.$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad y^3 - x = 0.$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1),$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$.

Next we calculate the second partial derivatives and $D(x,y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16.$$

Since $D(0,0) = -16 < 0$, it follows from case (c) of the Second Derivative Test that the origin is a saddle point; hence, f has no local maximum or minimum at $(0,0)$. Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see from case (a) of the test that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, so $f(-1,-1) = -1$ is also a local minimum value.

Definition 39 A subset $D \subset \mathbb{R}^2$ is **closed** if it contains all of its boundary points.

Definition 40 A subset $D \subset \mathbb{R}^2$ is **bounded** if it is contained within some disk in the plane.

Theorem 41 (Extreme Value Theorem for Functions of Two Variables) If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The next theorem is stated for a function f of three variables but there is a similar theorem for a function of two variables (see Example 43 below).

Theorem 42 (Method of Lagrange Multipliers) To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$):

1. Find all values of x, y, z, and λ such that
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
and
$$g(x, y, z) = k.$$
2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 43 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution : We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$, $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$2x = 2x\lambda \tag{2}$$
$$4y = 2y\lambda \tag{3}$$
$$x^2 + y^2 = 1. \tag{4}$$

From (2) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (4) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (3), so then (4) gives $x = \pm 1$. Therefore, f has possible extreme values at the points $(0, 1)$, $(0, -1)$ $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1.$$

Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$.

Example 44 Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Solution : We will compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0,0)$. We compare the value of f at that point with the extreme values on the boundary from Example 43:

$$f(0,0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2.$$

Therefore, the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0,0) = 0$.

We now start the second material for midterm 2 which concerns double integrals. For a positive, continuous function $f(x,y)$ defined on a closed and bounded domain $D \subset \mathbb{R}^2$, we denote by

$$\int\int_D f(x,y)\, dA,$$

the volume under the graph of $f(x,y)$ over D. This volume for a rectangle $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\} = [a,b] \times [b,c] \subset \mathbb{R}^2$ can be estimated by the following Midpoint Rule for Double Integrals described in the next theorem. We also use this rule for defining the double integral when $f(x,y)$ is not necessarily positive.

Theorem 45 (Midpoint Rule for Double Integrals) Let m, n be positive integers. Let $x_0 = a < x_1 < x_2 < \ldots < x_m = b$ be a division of $[a,b]$ into n intervals $[x_i, x_i + 1]$ of equal width $\Delta x = \frac{b-a}{m}$. Similarly, let $y_0 = c < y_1 < y_2 < \ldots < y_n = d$ be a division of $[c,d]$ into m intervals $[y_j, y_{j+1}]$ of equal widths $\Delta y = \frac{d-c}{n}$. Then:

$$\int\int_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\, \Delta A,$$

where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j}$ is the midpoint of $[y_{j-1}, y_j]$. Furthermore, the right-hand side above converges to the left-hand side as $m, n \to \infty$

Definition 46 If f is a continuous function of two variables, then its **average value** on a domain $D \subset \mathbb{R}^2$ is:

$$\frac{\int\int_D f(x,y)\, dA}{\text{Area}(D) = \int\int_D dA}.$$

Definition 47 The iterated integral of $f(x,y)$ on a rectangle $R = [a,b] \times [c,d]$ is

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \quad \text{or} \quad \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

One calculates the integral $\int_a^b \int_c^d f(x,y)\, dy\, dx$ by first calculating $A(x) = \int_c^d f(x,y)\, dy$, holding x constant, and then calculating $\int_a^b A(x)\, dx$ and similarly, for calculating the other integral.

Example 48 Evaluate the iterated integral.

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx$$

Solution : Regarding x as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2.$$

Thus, the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] \, dx = \int_0^3 \frac{3}{2} x^2 \, dx = \left. \frac{x^3}{2} \right]_0^3 = \frac{27}{2}.$$

Example 49 Evaluate the iterated integral.

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy.$$

Solution : Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] \, dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} \, dy = \int_1^2 9y \, dy = \left. 9\frac{y^2}{2} \right]_1^2 = \frac{27}{2}.$$

Theorem 50 (Fubini's Theorem) If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\}$, then

$$\int \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 51 Evaluate the double integral $\int\int_R (x - 3y^2) \ dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, \ 1 \leq y \leq 2\}$.

Solution : Fubini's Theorem gives

$$\int \int_R (x - 3y^2) \, dA = \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx = \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} \, dx$$

$$= \int_0^2 (x - 7) \, dx = \left. \frac{x^2}{2} - 7x \right]_0^2 = -12.$$

Example 52 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Solution : We first observe that S is the solid that lies under the surface $z = 16 - x^2 - y^2$ and above the square $R = [0, 2] \times [0, 2]$. We are now in a position to evaluate the double integral using Fubini's Theorem. Therefore,

$$V = \int\int_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$

$$= \int_0^2 [16x - \frac{1}{3}x^3 - 2y^2 x]_{x=0}^{x=2}\, dy$$

$$= \int_0^2 (\frac{88}{3}y - 4y^2)\, dy = \left[\frac{88}{3} - \frac{4}{3}y^3\right]_0^2 = 48.$$

In general, for any continuous function $f(x, y)$ on a closed and bounded domain $D \subset \mathbb{R}^2$, the integral $\int_D \int f(x, y)\, dA$ is defined and it is equal to the area under the graph of $f(x, y)$ on D when the function is positive. There are two cases for D, called type I and type II, where the integral

$$\int\int_D f(x, y)\, dA$$

can be calculated in a straightforward manner.

Definition 53 A plane region D is said to be of **type I**, if it can be expressed as

$$D = \{(x, y) \mid a \le x \le b, \;\; g_1(x) \le y \le g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous.

Definition 54 A plane region D is said to be of **type II**, if it can be expressed as

$$D = \{(x, y) \mid c \le y \le d, \; h_1(y) \le x \le h_2(y)\},$$

where h_1 and h_2 are continuous.

Theorem 55 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b, \;\; g_1(x) \le y \le g_2(x)\},$$

then

$$\int\int_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Theorem 56

$$\int\int_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

where D is a type II region given by Definition 54.

Example 57 Evaluate $\int\int_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution : The parabolas intersect when $2x^2 = 1 + x^2$, that is $x^2 = 1$, so $x = \pm 1$. We note that the region D, is a type I region but not a type II region and we can write

$$D = \{(x, y) - 1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}.$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Definition 53 gives

$$\int\int_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx = \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\,\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big|_{-1}^{1} = \frac{32}{15}.$$

Example 58 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution 1: We see that D is a type I region and

$$D = \{(x, y) \mid 0 \leq x \leq 2, \;\; x^2 \leq y \leq 2x\}.$$

Therefore, the volume under $z = x^2 + y^2$ and above D is

$$V = \int\int_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Bigg]_0^2 = \frac{216}{35}.$$

Solution 2: We see that D can also be written as a type II region:

$$D = \{(x, y) \mid 0 \leq y \leq 4, \;\; \frac{1}{2}y \leq x \leq \sqrt{y}\}.$$

Therefore, another expression for V is

$$V = \int\int_D (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \frac{2}{15}y^{\frac{5}{2}} + \frac{2}{7}y^{\frac{7}{2}} - \frac{13}{96}y^4\Bigg]_0^4 = \frac{216}{35}.$$

Example 59 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx$.

Solution : If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\ dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\ dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. We have

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx = \int\int_D \sin(y^2)\ dA,$$

where

$$D = \{(x, 0 \mid 0 \leq x \leq 1,\ \ x \leq y \leq 1)\}.$$

We see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leq y \leq 1,\ \ 0 \leq x \leq y\}.$$

This enables us to express the double integral as an iterated integral in the reverse order:

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx = \int\int_D \sin(y^2)\ dA = \int_0^1 \int_0^y \sin(y^2)\ dx\ dy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y}\ dy$$

$$= \int_0^1 y \sin(y^2)\ dy = -\frac{1}{2}\cos(y^2)\Big]_0^1 = \frac{1}{2}(1 - \cos 1).$$