Hi students!

I am putting this version of my review for the Final exam review here on the web site, place and time to be announced.
Enjoy!!
Best, **Bill Meeks**

PS. There are probably errors in some of the solutions presented here and for a few problems you need to complete them or simplify the answers; some questions are left to you the student. Also you might need to add more detailed explanations or justifications on the actual similar problems on your exam. I will keep updating these solutions with better corrected/improved versions. The first 6 slides are from Exam 2 practice problems but the material also falls on our Final exam.

Problem 23 - Exam 2 Fall 2006

Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$$

by reversing the order of integration.

Solution:

- Note that the region **R** defined by $\{(x,y) \mid \sqrt{y} \le x \le 1,\ \ 0 \le y \le 1\}$ is equal to the region $\{(x,y) \mid 0 \le x \le 1,\ \ 0 \le y \le x^2\}$.
- Thus,

$$\int\int_{\mathbf{R}} \sqrt{x^3+1}\, dA = \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy =$$

$$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 \left[\sqrt{x^3+1}\, y\right]_0^{x^2} dx = \int_0^1 \sqrt{x^3+1}\, x^2 dx$$

$$= \frac{1}{3}\int_0^1 (x^3+1)^{\frac{1}{2}} \cdot 3x^2\, dx = \frac{1}{3}\left(\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right)\Big|_0^1 = \frac{2}{9}(2^{\frac{3}{2}} - 1).$$

□

Problem 32(3) - From Exam 2

Find the iterated integral,

$$\int_0^1 \int_x^{2-x} (x^2 - y)dy\ dx.$$

Solution:

$$\int_0^1 \int_x^{2-x} (x^2 - y)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{y^2}{2}\right]_x^{2-x} dx$$

$$= \int_0^1 x^2(2-x) - \frac{(2-x)^2}{2} - (x^3 - \frac{x^2}{2})\,dx = 2\int_0^1 -x^3 + x^2 + x - 1\,dx$$

$$= -\frac{1}{2}x^4 + \frac{2}{3}x^3 + x^2 - 2x\Big|_0^1 = -\frac{1}{2} + \frac{2}{3} + 1 - 2 = -\frac{5}{6}.$$

$\square$

Problem 32(4) - From Exam 2

Find the iterated integral,

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy\ dx.$$

(Hint: Reverse the order of integration)

Solution:

- Reverse the order of integration:

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^4}{4} \sin(y^3)\right]_0^{\sqrt{y}} dy = \int_0^1 \frac{y^2}{4} \sin(y^3)\, dy.$$

- Let $u = y^3$ and then $du = 3y^2\, dy$. Making this substitution,

$$\int \frac{y^2}{4} \sin(y^3)\, dy = \frac{1}{12} \int \sin{(y^3)} 3y^2\, dy = -\frac{1}{12} \cos(y^3).$$

- Hence,

$$\int_0^1 \frac{y^2}{4} \sin(y^3)\, dy = -\frac{1}{12} \cos(y^3)\Big|_0^1 = \frac{1}{12}(1 - \cos(1)).$$

□

Problem 33(2) - From Exam 2

Evaluate the following double integral.

$$\int\int_{\mathbf{R}} e^{y^2} dA, \quad \mathbf{R} = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}.$$

Solution:

$$\int\int_{\mathbf{R}} e^{y^2}\, dA = \int_0^1 \int_0^y e^{y^2} dx\, dy = \int_0^1 \left[e^{y^2} x\right]_0^y\, dy$$

$$= \int_0^1 e^{y^2} y\, dy = \frac{1}{2}\int_0^1 e^{y^2}(2y)\, dy$$

$$= \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{1}{2}(e - e^0) = \frac{1}{2}(e-1).$$

$\square$

Problem 33(3) - From Exam 2

Evaluate the following double integral.

$$\int\int_{\mathbf{R}} x\sqrt{y^2 - x^2}dA, \quad \mathbf{R} = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}.$$

Solution:

$$\int\int_{\mathbf{R}} x\sqrt{y^2 - x^2}\, dA = \int_0^1 \int_0^y \sqrt{y^2 - x^2}\, x\, dx\, dy$$

$$= -\frac{1}{2}\int_0^1 \int_0^y (y^2 - x^2)^{\frac{1}{2}}(-2x)\, dx\, dy = -\frac{1}{2}\int_0^1 \frac{2}{3}(y^2 - x^2)^{\frac{3}{2}}\Big|_0^y\, dy$$

$$= \frac{1}{3}\int_0^1 y^3 dy = \frac{1}{3}\cdot\frac{1}{4}y^4\Big|_0^1 = \frac{1}{12}.$$

$\square$

Problem 34(2) - From Exam 2

Find the **volume V** of the solid under the surface $z = 2x + y^2$ and above the region bounded by curves $x - y^2 = 0$ and $x - y^3 = 0$.

Solution:

$$\mathbf{V} = \int_0^1 \int_{y^3}^{y^2} (2x + y^2)\, dx\, dy = \int_0^1 \left[x^2 + y^2 x \right]_{y^3}^{y^2} dy$$

$$= \int_0^1 y^4 + y^4 - (y^6 + y^2 y^3)\, dy = \int_0^1 -y^6 - y^5 + 2y^4\, dy$$

$$= -\frac{y^7}{7} - \frac{y^6}{6} + \frac{2}{5} y^5 \Big|_0^1 = -\frac{1}{7} - \frac{1}{6} + \frac{2}{5}.$$

□

Problem 1(a) - Fall 2008

Consider the points $A = (1,0,0)$, $B = (2,1,0)$ and $C = (1,2,3)$. Find the **parametric equations** for the line **L** passing through the points A and C.

Solution:

- A vector parallel to the line **L** is:

$$\mathbf{v} = \overrightarrow{AC} = \langle 1-1, 2-0, 3-0, \rangle = \langle 0,2,3 \rangle.$$

- A point on the line is $A = (1,0,0)$.
- Therefore **parametric equations** for the line **L** are:

$$\begin{aligned} x &= 1 \\ y &= 2t \\ z &= 3t. \end{aligned}$$

$\square$

Problem 1(b) - Fall 2008

Consider the points $A = (1, 0, 0)$, $B = (2, 1, 0)$ and $C = (1, 2, 3)$. Find an equation of the plane in $\mathbf{R}^3$ which contains the points A, B, C.

Solution:

Since a plane is determined by its normal vector $\mathbf{n}$ and a point on it, say the point A, it suffices to find $\mathbf{n}$. Note that:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = \langle 3, -3, 2 \rangle.$$

So the **equation of the plane** is:

$$\langle 3, -3, 2 \rangle \cdot \langle x - 1, y - 0, z - 0 \rangle = 3(x - 1) - 3y + 2z = 0.$$

□

Problem 1(c) - Fall 2008

Consider the points $A = (1,0,0)$, $B = (2,1,0)$ and $C = (1,2,3)$. Find the area of the triangle $\boldsymbol{\Delta}$ with vertices A, B and C.

Solution:

Consider the points $A = (1,0,0)$, $B = (2,1,0)$ and $C = (1,2,3)$. Then the area of the triangle $\boldsymbol{\Delta}$ with these vertices can be found by taking the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and dividing by 2. Thus:

$$\textbf{Area}(\boldsymbol{\Delta}) = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{1}{2}\left\| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{array} \right\|$$

$$= \frac{1}{2}|\langle 3, -3, 2\rangle| = \frac{1}{2}\sqrt{9+9+4} = \frac{1}{2}\sqrt{22}.$$

$\square$

Problem 2 - Fall 2008

Find the volume under the graph of $f(x, y) = x + 2xy$ and over the bounded region in the first quadrant $\{(x, y) \mid x \geq 0, \ \ y \geq 0\}$ bounded by the curve $y = 1 - x^2$ and the x and y-axes.

Solution:

$$\int_0^1 \int_0^{1-x^2} (x + 2xy)\; dy\; dx = \int_0^1 (xy + xy^2)\Big|_0^{1-x^2} \; dx$$

$$= \int_0^1 x(1-x^2) + x(1-x^2)^2 \; dx = \int_0^1 x^5 - 3x^3 + 2x \; dx$$

$$= (\frac{x^6}{6} - \frac{3x^4}{4} + x^2)\Big|_0^1 = \frac{1}{6} - \frac{3}{4} + 1 = \frac{5}{12}.$$

$\square$

Problem 3 - Fall 2008

Let

$$\mathbf{I} = \int_0^1 \int_{2x}^2 \sin(y^2)\, dy\, dx.$$

1. Sketch the region of integration.
2. Write the integral $\mathbf{I}$ with the order of integration reversed.
3. Evaluate the integral $\mathbf{I}$. Show your work.

Solution:

1. See the blackboard for a sketch.
2. $$\mathbf{I} = \int_0^1 \int_0^{\frac{1}{2}y} \sin(y^2)\, dx\, dy$$
3. By Fubini's Theorem,

$$\int_0^1 \int_0^{\frac{1}{2}y} \sin(y^2)\, dx\, dy = \int_0^1 \sin(y^2)x \Big|_0^{\frac{1}{2}y} dy = \int_0^1 (\sin(y^2) \cdot \frac{1}{2}y)\, dy$$

$$= \frac{1}{4}\int_0^1 \sin(y^2) \cdot 2y\, dy = \frac{\cos(y^2)}{4}\Big|_0^1 = \frac{1}{4}(\cos 1 - 1).$$

$\square$

Problem 4(a, b, c) - Fall 2008

Consider the function $\mathbf{F}(x, y, z) = x^2 + xy^2 + z$.

1. What is the gradient $\nabla \mathbf{F}(x, y, z)$ of $\mathbf{F}$ at the point $(1, 2, -1)$?
2. Calculate the directional derivative of $\mathbf{F}$ at the point $(1, 2, -1)$ in the direction $\langle 1, 1, 1 \rangle$?
3. What is the maximal rate of change of $\mathbf{F}$ at the point $(1, 2, -1)$?

Solution:

- $\nabla f = \langle 2x + y^2,\ 2xy,\ 1 \rangle$.
- So,

$$\nabla f(1, 2, -1) = \langle 6, 4, 1 \rangle.$$

- The unit vector $\mathbf{u}$ in the direction $\langle 1, 1, 1 \rangle$ is $\mathbf{u} = \frac{\langle 1,1,1 \rangle}{\sqrt{3}}$.
- $D_{\mathbf{u}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = \langle 6, 4, 1 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \frac{11}{\sqrt{3}}$.
- The **maximum rate of change** is the length of the **gradient**:

$$\mathbf{MRC}(f) = |\nabla f(1, 2, -1)| = |\langle 6, 4, 1 \rangle| = \sqrt{53}. \qquad \square$$

Problem 4(d) - Fall 2008

Consider the function $\mathbf{F}(x, y, z) = x^2 + xy^2 + z$. Find the equation of the tangent plane to the level surface $\mathbf{F}(x, y, z) = 4$ at the point $(1, 2, -1)$.

Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z) = x^2 + xy^2 + z$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$\nabla \mathbf{F}(x, y, z) = \langle 2x, 2xy, 1 \rangle$$

$$\mathbf{n} = \nabla \mathbf{F}(1, 2, -1) = \langle 6, 4, 1 \rangle.$$

- The equation of the **tangent plane** is:

$$\langle 6, 4, 1 \rangle \cdot \langle x-1, y-2, z+1 \rangle = 6(x-1) + 4(y-2) + (z+1) = 0.$$

$\square$

Problem 5 - Fall 2008

Find the volume **V** of the solid under the surface $z = 1 - x^2 - y^2$ and above the xy-plane.

Solution:

- The domain of integration for the function $z = 1 - x^2 - y^2$ described in polar coordinates is:

$$\mathbf{D} = \{(r,\theta) \mid 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}.$$

- In polar coordinates $r^2 = x^2 + y^2$ and so $z = 1 - r^2$.
- Applying Fubini's Theorem,

$$\begin{aligned} \mathbf{V} &= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2} r^2 - \frac{1}{4} r^4\right]_0^1 d\theta = \int_0^{2\pi} (\frac{1}{2} - \frac{1}{4}) \, d\theta \\ &= \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}. \end{aligned}$$

$\square$

Problem 6(a) - Fall 2008

Determine whether the following vector fields are **conservative** or not. Find a **potential function** for those which are indeed **conservative**.

1. $\mathbf{F}(x, y) = \langle x^2 + e^x + xy, xy - \sin(y)\rangle$.
2. $\mathbf{G}(x, y) = \langle 3x^2y + e^x + y^2, x^3 + 2xy + 3y^2\rangle$.

Solution:

On this slide we only consider the function $\mathbf{F}(x, y)$.

- Note that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y) = x^2 + e^x + xy$ and $Q(x, y) = xy - \sin(y)$.
- Since $P_y(x, y) = x \neq y = Q_x(x, y)$, the vector field $\mathbf{F}(x, y)$ is **not conservative**.

$\square$

Problem 6(b) - Fall 2008

Determine whether the following vector fields are **conservative** or not. Find a **potential function** for those which are indeed **conservative**.

1. $\mathbf{F}(x, y) = \langle x^2 + e^x + xy, xy - \sin(y) \rangle$.
2. $\mathbf{G}(x, y) = \langle 3x^2y + e^x + y^2, x^3 + 2xy + 3y^2 \rangle$.

Solution:

On this slide we only consider the function $\mathbf{G}(x, y)$.

- Since $\frac{\partial}{\partial y}(3x^2y + e^x + y^2) = 3x^2 + 2y = \frac{\partial}{\partial x}(x^3 + 2xy + 3y^2)$, there exists a **potential function** $\mathbf{f}(x, y)$, where $\nabla \mathbf{f} = \mathbf{G}$.
- Note that:
$$\frac{\partial \mathbf{f}}{\partial y} = x^3 + 2xy + 3y^2 \implies \mathbf{f}(x, y) = x^3y + xy^2 + y^3 + g(x),$$
  where $g(x)$ is some function of x.
- Since $\frac{\partial \mathbf{f}}{\partial x} = 3x^2y + e^x + y^2$,
$$\frac{\partial \mathbf{f}}{\partial x} = \frac{\partial(x^3y + xy^2 + y^3 + g(x))}{\partial x} = 3x^2y + y^2 + g'(x) = 3x^2y + e^x + y^2$$
$$\implies g'(x) = e^x \implies g(x) = e^x + \text{constant}.$$
- Hence, $\mathbf{f}(x, y) = x^3y + xy^2 + y^3 + e^x$ is a **potential function**. □

Problem 7 - Fall 2008

Evaluate the line integral

$$\int_{\mathbf{C}} yz\ dx + xz\ dy + xy\ dz,$$

where $\mathbf{C}$ is the curve starting at $(0,0,0)$, traveling along a line segment to $(1,0,0)$ and then traveling along a second line segment to $(1,2,1)$.

Solution:

- The parameterizations $\mathbf{C}_1(t), \mathbf{C}_2(t)$ are:
  $\mathbf{C}_1(t) = \langle t, 0, 0 \rangle \qquad 0 \le t \le 1$
  $\mathbf{C}_2(t) = \langle 1, 2t, t \rangle \qquad 0 \le t \le 1$
- So, $\mathbf{C}_1'(t) = \langle 1, 0, 0 \rangle$ and $\mathbf{C}_2(t) = \langle 0, 2, 1 \rangle$.
- Thus, $\int_{\mathbf{C}_1} yz\ dx + xy\ dy + xy\ dz =$
  $\int_0^1 [(0 \cdot 0 \cdot 1) + (t \cdot 0 \cdot 0) + (t \cdot 0 \cdot 0)]\ dt = 0.$
- Also, $\int_{\mathbf{C}_2} yz\ dx + xz\ dy + xy\ dz$
  $= \int_0^1 [(2t \cdot t \cdot 0) + (1 \cdot t \cdot 2) + (1 \cdot 2t \cdot 1)]\ dt = \int_0^1 4t\ dt = \left. \frac{4t^2}{2} \right|_0^1$
  $= \frac{4}{2} = 2.$ So, the entire integral equals $0 + 2 = 2$.

Problem 8(a) - Fall 2008

Use Green's Theorem to show that if $\mathbf{D} \subset \mathbf{R}^2$ is the bounded region with boundary a positively oriented simple closed curve $\mathbf{C}$, then the area of $\mathbf{D}$ can be calculated by the formula:

$$\mathbf{Area}(\mathbf{D}) = \frac{1}{2} \oint_{\mathbf{C}} -y\, dx + x\, dy$$

Solution:

- Recall Green's Theorem:

$$\oint_{\mathbf{C}} P\, dx + Q\, dy = \int\int_{\mathbf{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

- Hence,

$$\frac{1}{2} \oint_{\mathbf{C}} -y\, dx + x\, dy = \int\int_{\mathbf{D}} \left(\frac{\partial x}{\partial x} - \frac{\partial - y}{\partial y}\right) dA$$

$$= \frac{1}{2} \int\int_{\mathbf{D}} (1+1)\, dA = \int\int_{\mathbf{D}} dA = \mathbf{Area}(\mathbf{D})$$

$\square$

Problem 8(b) - Fall 2008

Consider the ellipse $4x^2 + y^2 = 1$. Use the above area formula to calculate the area of the region $\mathbf{D} \subset \mathbf{R}^2$ with boundary this ellipse. (Hint: This ellipse can be parametrized by $\mathbf{r}(t) = \langle \frac{1}{2}\cos(t), \sin(t)\rangle$ for $0 \leq t \leq 2\pi$.)

Solution:

- The ellipse has **parametric equations** $x = \frac{1}{2}\cos t$ and $y = \sin t$, where $0 \leq t \leq 2\pi$.
- Using the formula in the previous theorem, we have:

$$\begin{aligned}
\mathbf{Area}(\mathbf{D}) &= \frac{1}{2}\oint_{\mathbf{C}} x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (\frac{1}{2}\cos t)(\cos t)\,dt - (\sin t)(-\frac{1}{2}\sin t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{2}(\cos^2 t + \sin^2 t)\,dt = \frac{2\pi}{4} = \frac{\pi}{2}.
\end{aligned}$$

$\square$

Problem 9(a) - Spring 2008

For the space curve $\mathbf{r}(t) = \langle t^2 - 1, t^2, t/2 \rangle$,

(a) find the velocity, speed, and acceleration of a particle whose position function is $\mathbf{r}(t)$ at time $t = 4$.

Solution:

- The velocity $\mathbf{v}(t)$ is equal to $\mathbf{r}'(t)$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 2t, \frac{1}{2} \rangle$$
$$\mathbf{v}(4) = \langle 8, 8, \frac{1}{2} \rangle.$$

- The acceleration $\mathbf{a}(t)$ is equal to $\mathbf{v}'(t)$:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 0 \rangle$$
$$\mathbf{a}(4) = \langle 2, 2, 0 \rangle.$$

- The speed $\mathbf{s}(4)$ is equal to $|\mathbf{v}(4)|$:

$$\mathbf{s}(4) = |\mathbf{v}(4)| = \sqrt{64 + 64 + \frac{1}{4}} = \sqrt{128 + \frac{1}{4}}.$$

□

Problem 9(b) - Spring 2008

For the space curve $\mathbf{r}(t) = \langle t^2 - 1, t^2, t/2\rangle$,

(b) find all points where the particle with position vector $\mathbf{r}(t)$ intersects the plane $x + y - 2z = 0$.

Solution:

- Plug the x, y and z-coordinates of $\mathbf{r}(t)$ into the equation of the plane and solve for t:

$$(t^2 - 1) + t^2 - 2(\frac{t}{2}) = 2t^2 - t - 1 = (2t + 1)(t - 1) = 0$$

$$\implies t = 1 \text{ or } \quad t = -\frac{1}{2}.$$

- Next evaluate the points on $\mathbf{r}(t)$ at these times to obtain the 2 points of intersection:

$$\mathbf{r}(1) = \langle 1^2 - 1, 1^2, \frac{1}{2}\rangle = \langle 0, 1, \frac{1}{2}\rangle$$

$$\mathbf{r}(-\frac{1}{2}) = \langle(-\frac{1}{2})^2 - 1, (-\frac{1}{2})^2, \frac{1}{2}(-\frac{1}{2})\rangle = \langle -\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}\rangle. \quad \square$$

Problem 10 - Spring 2008

Let **D** be the region of the xy-plane above the graph of $y = x^2$ and below the line $y = x$.

(a) Determine an iterated integral expression for the double integral $\int\int_{\mathbf{D}} xy\, dA$

(b) Find an equivalent iterated integral to the one found in part (a) with the reversed order of integration.

(c) Evaluate one of the two iterated integrals in parts (a), (b).

Solution:

- Part (a)
$$\int\int_{\mathbf{D}} xy\ dA = \int_0^1 \int_{x^2}^{x} xy\ dy\, dx.$$

- Part (b)
$$\int_0^1 \int_y^{\sqrt{y}} xy\, dx\, dy.$$

- Part (c)
$$\int_0^1 \int_{x^2}^{x} xy\, dy\, dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_{x^2}^{x} dx$$
$$= \int_0^1 \left(\frac{1}{2}x^3 - \frac{1}{2}x^5\right) dx = \frac{1}{8}x^4 - \frac{1}{12}x^6\Big|_0^1 = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}$$

□

Problem 11 - Spring 2008

Find the absolute **maximum** and absolute **minimum** values of $f(x,y) = x^2 + 2y^2 - 2y$ in the set $\mathbf{D} = \{(x,y) : x^2 + y^2 \leq 4\}$.

Solution:

- Find critical points of $f(x,y)$:

$$\nabla f = \langle 2x, 4y-2\rangle = 0 \Longrightarrow x = 0 \text{ and } y = \frac{1}{2}.$$

- Use **Lagrange multipliers** to study max and min values of f on the circle $g(x,y) = x^2 + y^2 = 4$:

$$\nabla f \langle 2x, 4y-2\rangle = \lambda \nabla g = \lambda\langle 2x, 2y\rangle.$$

- We get

$$2x = \lambda 2x \Longrightarrow \lambda = 1 \text{ or } x = 0.$$

- If $\lambda = 1$, then $4y - 2 = 2y \Longrightarrow y = 1$.
- Plugging in $g(x,y)$, gives $(0, \frac{1}{2})$, $(0, \pm 2)$ and $(\pm\sqrt{3}, 1)$. We get $f(0, \frac{1}{2}) = -\frac{1}{2}$, $f(0,2) = 4$, $f(0,-2) = 12$, $f(\pm\sqrt{3}, 1) = 3$.
- The **maximum** value of $f(x,y)$ is 12 and its **minimum** value is $-\frac{1}{2}$.

$\square$

Problem 12(a) - Spring 2008

Let **D** be the region in the first quadrant $x, y \geq 0$ that lies between the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(a) Describe the region **D** using polar coordinates.

(b) Evaluate the double integral $\int \int_{\mathbf{D}} 3x + 3y \, dA$.

Solution:

- The domain is: $$\mathbf{D} = \{(r, \theta) \mid 2 \leq r \leq 3, \;\; 0 \leq \theta \leq \frac{\pi}{2}\}.$$
- Calculate the integral using the substitution:

$$x = r\cos\theta \quad y = r\sin\theta \quad dA = r\,dr\,d\theta;$$

$$\int\int_{\mathbf{D}} (3x + 3y) dA = \int_0^{\frac{\pi}{2}} \int_2^3 (3r\cos\theta + 3r\sin\theta) r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_2^3 3r^2(\cos\theta + \sin\theta)\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[r^3(\cos\theta + \sin\theta)\right]_2^3 d\theta$$

$$= \int_0^{\frac{\pi}{2}} (27 - 8)(\cos\theta + \sin\theta)\,d\theta = \; 19(\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}} = 19 + 19 = 38. \quad \square$$

Problem 13(a) - Spring 2008

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1, 0, -1)$ for the implicit function z
determined by the equation $x^3 + y^3 + z^3 - 3xyz = 0$.

Solution:

- Consider the function $\mathbf{F}(x, y, z) = x^3 + y^3 + z^3 - 3yxz$.
- Since $\mathbf{F}(x, y, z)$ is constant on the surface, the **Chain Rule** gives:

$$\frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial \mathbf{F}}{\partial y}\frac{\partial x}{\partial y} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial y} = 0.$$

- Thus,

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} = \frac{-3x^2 + 3yz}{3z^2 - 3xy} = \frac{-x^2 + yz}{z^2 - xy},$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} = \frac{-3y^2 + 3xz}{3z^2 - 3xy} = \frac{-y^2 + xz}{z^2 - xy}.$$

$\square$

Problem 13(b) - Spring 2008

(b) Is the **tangent plane** to the surface $x^3 + y^3 + z^3 - 3xyz = 0$ at the point $(1, 0, -1)$ **perpendicular** to the plane $2x + y - 3z = 2$? Justify your answer with an appropriate calculation.

Solution:

- Since $\mathbf{F}(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ is constant along the surface $\mathbf{F}(x, y, z) = 0$, $\nabla\mathbf{F}$ is normal (orthogonal) to the surface.
- Calculating, we obtain:

$$\nabla\mathbf{F} = \langle 3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy \rangle$$

$$\nabla\mathbf{F}(1, 0, -1) = \langle 3, 3, 3 \rangle,$$

  which is the normal vector to the **tangent plane** of the surface.
- Since the normal to the plane $2x + y - 3z = 2$ is $\langle 2, 1, -3 \rangle$ and $\langle 3, 3, 3 \rangle \cdot \langle 2, 1, -3 \rangle = 0$, it is **perpendicular**. $\square$

Problem 14(a) - Spring 2008

(a) Consider the vector field $\mathbf{G}(x, y) = \langle 4x^3 + 2xy, x^2 \rangle$. Show that $\mathbf{G}$ is *conservative* (i.e. $\mathbf{G}$ is a *potential* or a *gradient* vector field), and use the fundamental theorem for line integrals to determine the value of $\int_{\mathbf{C}} \mathbf{G} \cdot \mathbf{dr}$, where $\mathbf{C}$ is the contour consisting of the line starting at $(2, -2)$ and ending at $(-1, 1)$.

Solution:

- Since $\frac{\partial}{\partial y}(4x^3 + 2xy) = 2x = \frac{\partial}{\partial x}(x^2)$, there exists a **potential function** $\mathbf{F}(x, y)$, where $\nabla \mathbf{F} = \mathbf{G}$.
- Note that:
$$\frac{\partial \mathbf{F}}{\partial y} = x^2 \Longrightarrow \mathbf{F}(x, y) = x^2 y + g(x),$$
where $g(x)$ is some function of x.
- Since $\frac{\partial \mathbf{F}}{\partial x} = 4x^3 + 2xy$,
$$\frac{\partial \mathbf{F}}{\partial x} = \frac{\partial (x^2 y + g(x))}{\partial x} = 2xy + g'(x) = 4x^3 + 2xy$$
$$\Longrightarrow g'(x) = 4x^3 \Longrightarrow g(x) = x^4 + \text{constant}.$$
- Hence, $\mathbf{F}(x, y) = x^4 + x^2 y$ is a **potential function**.
- By the fundamental theorem of calculus for line integrals,
$$\int_{\mathbf{C}} \mathbf{G} \cdot \mathbf{dr} = \int_{\mathbf{C}} \nabla \mathbf{F} \cdot \mathbf{dr} = \mathbf{F}(-1, 1) - \mathbf{F}(2, -2) = 2 - 8 = -6.$$

$\square$

Problem 14(b) - Spring 2008

(b) Now let $\mathbf{T}$ denote the closed contour consisting of the triangle with vertices at (0,0),(1,0), and (1,1) with the counterclockwise orientation, and let $\mathbf{F}(x,y) = \langle \frac{1}{2}y^2 - y, xy \rangle$. Compute $\int_{\mathbf{T}} \mathbf{F} \cdot \mathbf{dr}$ directly (from the definition of line integral).

Solution:

- The curve $\mathbf{T}$ is the union of the segment $\mathbf{C}_1$ from $(0,0)$ to $(1,0)$, the segment $\mathbf{C}_2$ from $(1,0)$ to $(1,1)$ and the segment $\mathbf{C}_3$ from $(1,1)$ to $(0,0)$.
- Parameterize these segments as follows:

$$\begin{array}{ll} \mathbf{C}_1(t) = \langle t, 0 \rangle & 0 \le t \le 1 \\ \mathbf{C}_2(t) = \langle 1, t \rangle & 0 \le t \le 1 \\ \mathbf{C}_2(t) = \langle 1-t, 1-t \rangle & 0 \le t \le 1 \end{array}$$

- $$\boxed{\int_{\mathbf{T}} \mathbf{F} \cdot \mathbf{dr} = \int_{\mathbf{C}_1} \mathbf{F} \cdot \mathbf{dr} + \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dr} + \int_{\mathbf{C}_3} \mathbf{F} \cdot \mathbf{dr}}$$

$$= \int_0^1 \langle 0,0 \rangle \cdot \langle 1,0 \rangle \, dt + \int_0^1 \langle \frac{1}{2}t^2 - t, t \rangle \cdot \langle 0,1 \rangle \, dt + \int_0^1 \left\langle \frac{1}{2}(1-t)^2 - (1-t), (1-t)^2 \right\rangle \cdot \langle -1, -1 \rangle \, dt$$

$$= 0 + \int_0^1 t \, dt + \int_0^1 -\frac{3}{2}t^2 + 2t - \frac{1}{2} \, dt = \int_0^1 -\frac{3}{2}t^2 + 3t - \frac{1}{2} \, dt$$

$$= -\frac{1}{2}t^3 + \frac{3}{2}t^2 - \frac{1}{2}t \Big|_0^1 = -\frac{1}{2} + \frac{3}{2} - \frac{1}{2} = \frac{1}{2}. \qquad \square$$

Problem 14(c) - Spring 2008

Let $\mathbf{F}(x, y) = \langle \frac{1}{2}y^2 - y, xy \rangle$.

(c) Explain how Green's theorem can be used to show that the integral $\int_{\mathbf{T}} \mathbf{F} \cdot d\mathbf{r}$ in (b) must be equal to the area of the region **D** interior to **T**.

Solution:

- By Green's Theorem,

$$\int_{\mathbf{T}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{T}} (\frac{1}{2}y^2 - y)\, dx + xy\, dy$$

$$= \int\int_{\mathbf{D}} \frac{\partial(xy)}{\partial x} - \frac{\partial(\frac{1}{2}y^2 - y)}{\partial y}\, dA = \int\int_{\mathbf{D}} (y - y + 1)\, dA$$

$$= \int\int_{\mathbf{D}} dA.$$

- Since $\frac{1}{2} = \int\int_{\mathbf{D}} dA$ is the area of the triangle **D**, then the integral $\frac{1}{2}$ in part (b) is equal to this area. $\square$

Problem 15(a,b,c) - Fall 2007

Let
$$\mathbf{I} = \int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy.$$

(a) Sketch the region of integration

(b) Write the integral $\mathbf{I}$ with the order of integration reversed.

(c) Evaluate the integral $\mathbf{I}$. Show your work.

Solution:

- (a) (b) Make your sketch and from the sketch, we see:
$$\mathbf{I} = \int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy = \int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx.$$
- (c) We next evaluate the integral using part (b).
$$\mathbf{I} = \int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^2 \left[e^{x^3} y\right]_0^{x^2} dx = \int_0^2 e^{x^3} x^2\, dx$$
$$= \frac{1}{3}\int_0^2 e^{x^3}(3x^2)\, dx = \frac{1}{3} e^{x^3}\Big|_0^2 = \frac{1}{3}(e^8 - e^0) = \frac{1}{3}(e^8 - 1). \qquad \square$$

Problem 16 - Fall 2007

Find the distance from the point $(3, 2, -7)$ to the line **L**

$$x = 1 + t, \quad y = 2 - t, \quad z = 1 + 3t.$$

Solution:

- Note that the plane **T** passing through $P = (3, 2, -7)$ with normal vector the direction $\mathbf{n} = \langle 1, -1, 3 \rangle$ of the line **L** must intersect **L** in the point Q closest to P. We now find Q.
- The equation of the plane **T** is:

$$(x - 3) - (y - 2) + 3(z + 7) = 0.$$

- Substitute in this equation the parametric values of **L** and solve for t:

$$0 = (1 + t) - 3 - [(2 - t) - 2] + 3[(1 + 3t) + 7] = 11t + 22.$$

- Hence, $t = -2$ and $Q = \langle -1, 4, -5 \rangle$.
- The distance from P and Q is $\mathbf{d} = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$, and thus the distance from P to **L**. $\square$

Problem 17(a) - Fall 2007

Find the velocity and acceleration of a particle moving along the curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

at the point $(2, 4, 8)$.

Solution:

- Recall that the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration $\mathbf{a}(t) = \mathbf{r}'(t)$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, 6t \rangle.$$

- As the point $(2, 4, 8)$ corresponds to $t = 2$ on $\mathbf{r}(t)$,

$$\mathbf{v}(2) = \langle 1, 4, 12 \rangle$$
$$\mathbf{a}(2) = \langle 0, 2, 12 \rangle.$$

□

Problem 17(b) - Fall 2007

Find all points where the curve in part (a) intersects the surface $z = 3x^3 + xy - x$.

Solution:

- Plug the x, y and z-coordinates into the equation of the surface and solve for t.

$$t^3 = 3t^3 + t^3 - t$$

$$\implies 3t^3 - t = t(3t^2 - 1) = 0 \implies t = 0$$

or
$$t = \pm\frac{1}{\sqrt{3}}.$$

- Next plug these t values into $\mathbf{r}(t)$ to get the 3 points of intersection:

$$\mathbf{r}(0) = \langle 0, 0, 0, \rangle$$

$$\mathbf{r}(\frac{1}{\sqrt{3}}) = \langle \frac{1}{\sqrt{3}}, \frac{1}{3}, \frac{1}{3\sqrt{3}} \rangle$$

$$\mathbf{r}(-\frac{1}{\sqrt{3}}) = \langle -\frac{1}{\sqrt{3}}, \frac{1}{3}, -\frac{1}{3\sqrt{3}} \rangle.$$

$\square$

Problem 18 - Fall 2007

Find the **volume V** of the solid which lies below the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 3$.

Solution:

- We first describe in polar coordinates the domain $\mathbf{D} \subset \mathbf{R}^2$ for the integral. $\mathbf{D} = \{(r, \theta) \mid 0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi\}$.
- The graphing function is $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$.
- This gives the volume:

$$\mathbf{V} = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4 - r^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4 - r^2}\, r\, dr\, d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_0^{\sqrt{3}} (4 - r^2)^{\frac{1}{2}} (-2r)\, dr\, d\theta = -\frac{1}{2} \int_0^{2\pi} \left[\frac{2}{3}(4 - r^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}} d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} (1 - 8)\, d\theta = \int_0^{2\pi} \frac{7}{3}\, d\theta = \frac{7}{3}\theta \Big|_0^{2\pi} = \frac{14\pi}{3}. \quad \square$$

Problem 19(a) - Fall 2007

Consider the line integral $$\int_C \sqrt{1+x}\,dx + 2xy\,dy,$$ where **C** is the triangular path starting from $(0,0)$, to $(2,0)$, to $(2,3)$, and back to $(0,0)$.

(a) Evaluate this line integral directly, **without** using Green's Theorem.

Solution:

- The curve **C** is the union of the segment $\mathbf{C}_1$ from $(0,0)$ to $(2,0)$, the segment $\mathbf{C}_2$ from $(2,0)$ to $(2,3)$ and the segment $\mathbf{C}_3$ from $(2,3)$ to $(0,0)$.
- Parameterize these segments:

$$\begin{aligned} \mathbf{C}_1(t) &= \langle 2t, 0\rangle & 0 \le t \le 1 \\ \mathbf{C}_2(t) &= \langle 2, 3t\rangle & 0 \le t \le 1 \\ \mathbf{C}_3(t) &= \langle 2-2t, 3-3t\rangle & 0 \le t \le 1. \end{aligned}$$

- Thus, $\int_C \sqrt{1+x}\,dx + 2xy\,dy$

$$= \int_{C_1} \sqrt{1+x}\,dx + 2xy\,dy + \int_{C_2} \sqrt{1+x}\,dx + 2xy\,dy + \int_{C_3} \sqrt{1+x}\,dx + 2xy\,dy$$

$$= \int_0^1 \sqrt{1+2t}(2)\,dt + \int_0^1 4\cdot 3t(3)\,dt + \int_0^1 \sqrt{1+(2-2t)}(-2) + 2(2-2t)(3-3t)(-3)dt$$

$$= \int_0^1 \left(2\sqrt{1+2t} + 36t^2 + \sqrt{3-2t} - 36t + 72t - 36\right) dt.$$

- This long straightforward integral is left to you the student to do. □

Problem 19(b) - Fall 2007

Consider the line integral

$$\int_{\mathbf{C}} \sqrt{1+x}\, dx + 2xy\, dy,$$

where **C** is the triangular path starting from $(0,0)$, to $(2,0)$, to $(2,3)$, and back to $(0,0)$.

Evaluate this line integral using Green's theorem.

Solution:

- Let **D** denote the **2**-dimensional triangle bounded by **C**.
- Green's Theorem gives:

$$\int_{\mathbf{C}} \sqrt{1+x}\, dx + 2xy\, dy = \int\int_{\mathbf{D}} \left(\frac{\partial(2xy)}{\partial x} - \frac{\partial(\sqrt{1+x})}{\partial y} \right) dA$$

$$= \int\int_{\mathbf{D}} 2y\, dA = \int_0^2 \int_0^{\frac{3}{2}x} 2y\, dy\, dx = \int_0^2 \left[y^2 \right]_0^{\frac{3}{2}x} dx$$

$$= \int_0^2 \frac{9}{4}x^2\, dx = \frac{3}{4}x^3 \Big|_0^2 = 6.$$

$\square$

Problem 20(a) - Fall 2007

Consider the vector field $\mathbf{F} = (y^2/x^2)\mathbf{i} - (2y/x)\mathbf{j}$.
Find a function f such that $\nabla f = \mathbf{F}$.

Solution:

- Suppose f exists. Then:

$$\frac{\partial f}{\partial x} = \frac{y^2}{x^2} \Longrightarrow f(x,y) = \int \frac{y^2}{x^2}\,dx = -\frac{y^2}{x} + g(y),$$

  where $g(y)$ is a function of y.
- Then:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(-\frac{y^2}{x} + g(y)) = -\frac{2y}{x} + g'(y) = -\frac{2y}{x}.$$

- Hence,

$$g'(y) = 0 \Longrightarrow g(y) \text{ is constant.}$$

- Taking the constant $g(y)$ to be zero, we obtain:

$$f(x,y) = -\frac{y^2}{x}.$$

□

Problem 20(b) - Fall 2007

Consider the vector field $\mathbf{F} = (y^2/x^2)\mathbf{i} - (2y/x)\mathbf{j}$.
Let $\mathbf{C}$ be the curve given by $\mathbf{r}(t) = \langle t^3, \sin t\rangle$ for $\frac{\pi}{2} \leq t \leq \pi$.
Evaluate the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$.

Solution:

- We will apply the **potential function** $f(x, y) = \frac{-y^2}{x}$ in part (a) and the fundamental theorem of calculus for line integrals.
- We get:

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(\frac{\pi}{2})) = f(\pi^3, 0) - f(\frac{\pi^3}{8}, 1) = \frac{8}{\pi^3} \qquad \square$$

Problem 21 - Fall 2006

Find **parametric equations** for the line **L** in which the planes $x - 2y + z = 1$ and $2x + y + z = 1$ intersect.

Solution:

- The direction vector $\mathbf{v}$ of the line **L** is parallel to both planes.
- Hence, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ where $\mathbf{n}_1 = \langle 1, -2, 1 \rangle$ and $\mathbf{n}_2 = \langle 2, 1, 1 \rangle$ are the normal vectors of the respective planes:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \langle -3, 1, 5 \rangle.$$

- Next find the intersection point of **L** with the xy-plane by setting $z = 0$:

$$x - 2y = 1$$
$$2x + y = 1$$

$\Longrightarrow (\frac{3}{5}, -\frac{1}{5}, 0)$ is the intersection point.

- **Parametric equations** are:

$$\begin{aligned} x &= \tfrac{3}{5} - 3t \\ y &= -\tfrac{1}{5} + t \\ z &= 5t. \end{aligned}$$

□

Problem 22 - Fall 2006

Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1,1,1)$ which lies on the surface.

(i) Find the equation of the **tangent plane** to the surface at P.

(ii) Find the equation of the **normal line** to the surface at P.

Solution:

- Recall that the gradient of $\mathbf{F}(x,y,z) = x^2 + y^2 - 2z^2$ is normal (orthogonal) to the surface.
- Calculating, we obtain:

$$\nabla\mathbf{F}(x,y,z) = \langle 2x, 2y, -4z\rangle$$
$$\nabla\mathbf{F}(1,1,1) = \langle 2, 2, -4\rangle.$$

- The equation of the **tangent plane** is:

$$\langle 2,2,-4\rangle \cdot \langle x-1, y-1, z-1\rangle = 2(x-1)+2(y-1)-4(z-1) = 0$$

- The **vector equation** of the **normal line** is:

$$\mathbf{L}(t) = \langle 1,1,1,\rangle + t\langle 2,2,-4\rangle = \langle 1+2t, 1+2t, 1-4t\rangle. \quad \square$$

Problem 23 - Fall 2006

Find the **maximum** and **minimum** values of the function

$$f(x, y) = x^2 + y^2 - 2x$$

on the disc $x^2 + y^2 \leq 4$.

Solution:

- We first find the critical points.

$$\nabla f = \langle 2x - 2, 2y \rangle = 0 \Longrightarrow x = 1 \text{ and } y = 0.$$

- Next use **Lagrange multipliers** to study max and min of f on the boundary circle $g(x, y) = x^2 + y^2 = 4$:
  $\nabla f = \langle 2x - 2, 2y \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$.
- $2y = \lambda 2y \Longrightarrow y = 0$ or $\lambda = 1$.
- $y = 0 \Longrightarrow x = \pm 2$.
- $\lambda = 1 \Longrightarrow 2x - 2 = 2x$, which is impossible.
- Now check the values of f at 3 points:

$$f(1, 0) = -1, \quad f(2, 0) = 0, \quad f(-2, 0) = 8.$$

- The **maximum** value is 8 and the **minimum** value is -1. $\square$

Problem 24 - Fall 2006

Evaluate the iterated integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

Solution:

- The domain of integration for the function described in polar coordinates is:
$$\mathbf{D} = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}.$$
- Since
$$dA = dy\, dx$$
in polar coordinates is: $r\, dr\, d\theta$ and $r = \sqrt{x^2+y^2}$,
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2\, dr\, d\theta$$
$$\int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^1 d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta = \frac{\theta}{3}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{6}.$$

□

Problem 25(b) - Fall 2006

Find the **volume V** of the solid under the surface $z = 4 - x^2 - y^2$ and above the region in the xy-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

- The domain of integration for the function $z = 4 - x^2 - y^2$ described in polar coordinates is:

$$\mathbf{D} = \{(r, \theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi\}.$$

- In polar coordinates $r^2 = x^2 + y^2$ and so $z = 4 - r^2$.
- Applying Fubini's Theorem,

$$\mathbf{V} = \int_0^{2\pi} \int_1^2 (4 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 (4r - r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_1^2 d\theta = \int_0^{2\pi} (8 - 4) - (2 - \frac{1}{4}) d\theta$$

$$= \frac{9}{4} \int_0^{2\pi} d\theta = \frac{9\pi}{2}.$$

□

Problem 26(a) - Fall 2006

Determine whether the following vector fields are conservative or not. Find a **potential function** for those which are indeed conservative.

(a) $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (xy - y^2)\mathbf{j}$.

Solution:

- Note that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y) = x^2 + xy$ and $Q(x, y) = xy - y^2$.
- Since $P_y(x, y) = x \neq y = Q_x(x, y)$, the vector field $\mathbf{F}(x, y)$ is **not** conservative.

$\square$

Problem 26(b) - Fall 2006

Determine whether the following vector fields are conservative or not. Find a **potential function** for those which are indeed conservative.

(b) $\mathbf{F}(x,y) = (3x^2y + y^2)\mathbf{i} + (x^3 + 2xy + 3y^2)\mathbf{j}$.

Solution:

- Note that $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, where $P(x,y) = 3x^2y + y^2$ and $Q(x,y) = x^3 + 2xy + 3y^2$.
- Since $P_y(x,y) = 3x^2 + 2y = Q_x(x,y)$ and $P(x,y)$ and $Q(x,y)$ are infinitely differentiable on the entire plane, $\mathbf{F}(x,y)$ has a **potential function** $f(x,y)$. Thus,
- $$\frac{\partial f}{\partial x} = 3x^2y + y^2 \implies f(x,y) = \int 3x^2y + y^2 \, dx = x^3y + y^2x + g(y).$$
- Since $f_y = Q$, then
  $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y + y^2x + g(y)) = x^3 + 2xy + g'(y) = x^3 + 2xy + 3y^2.$$
- Hence, $g'(y) = 3y^2 \implies g(y) = y^3 + \mathbf{C}$, for some constant $\mathbf{C}$.
- Taking $\mathbf{C} = 0$, gives: $\quad f(x,y) = x^3y + y^2x + y^3$. $\square$

Problem 27 - Fall 2006

Evaluate the line integral $\int_{\mathbf{C}}(x^2+y)dx+(xy+1)dy$, where $\mathbf{C}$ is the curve starting at $(0,0)$, traveling along a line segment to $(1,2)$ and then traveling along a second line segment to $(0,3)$.

Solution:

- Let $\mathbf{C}_1$ be the segment from $(0,0)$ to $(1,2)$ and let $\mathbf{C}_2$ be the segment from $(1,2)$ to $(0,3)$.
- Parameterizations for these segments are:

$$\mathbf{C}_1(t)=\langle t,2t\rangle \qquad 0\le t\le 1$$
$$\mathbf{C}_2(t)=\langle 1-t,2+t\rangle \qquad 0\le t\le 1.$$

- Now calculate:

$$\int_{\mathbf{C}}(x^2+y)\,dx+(xy+1)\,dy=\int_{\mathbf{C}_1\cup\mathbf{C}_2}(x^2+y)\,dx+(xy+1)\,dy$$
$$=\int_0^1(t^2+2t)\,dt+(2t^2+1)2\,dt+\int_0^1[(1-t)^2+2+t](-1)+(1-t)(2+t)+1\,dt$$
$$=\int_0^1 3t^2+2t-2\,dt=\left.t^3+t^2+2t\right|_0^1=1+1+2=4.$$

□

Problem 28 - Fall 2006

Use Green's Theorem to evaluate the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle y^3 + \sin 2x, 2xy^2 + \cos y \rangle$ and $\mathbf{C}$ is the unit circle $x^2 + y^2 = 1$ which is oriented counterclockwise.

Solution:

- First rewrite $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ in standard form:

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} (y^3 + \sin 2x)\, dx + (2xy^2 + \cos y)\, dy.$$

- Recall and apply Green's Theorem:

$$\int_{\mathbf{C}} P\, dx + Q\, dy = \int\int_{\mathbf{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA,$$

where $\mathbf{D}$ is the disk with boundary $\mathbf{C}$:

$$\int_{\mathbf{C}} (y^3 + \sin 2x)\, dx + (2xy^2 + \cos y)\, dy = \int\int_{\mathbf{D}} 2y^2 - 3y^2 dA = -\int\int_{\mathbf{D}} y^2\, dA.$$

- Next evaluate the integral using polar coordinates:

$$-\int\int_{\mathbf{D}} y^2\, dA = -\int_0^{2\pi}\int_0^1 (r\sin\theta)^2 r\, dr\, d\theta = -\int_0^{2\pi} \left[\frac{r^4}{4}\sin^2\theta\right]_0^1 d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi} \sin^2\theta\, d\theta.$$

□

Problem 29

(a) Express the double integral $\int\int_{\mathbf{R}} x^2y - x \, dA$ as an iterated integral and evaluate it, where **R** is the first quadrant region enclosed by the curves $y = 0$, $y = x^2$ and $y = 2 - x$.

Solution:

- First rewrite the integral as an iterated integral.

$$\int\int_{\mathbf{R}} (x^2y - x) \, dA = \int_0^1 \int_{\sqrt{y}}^{2-y} (x^2y - x) \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^3y^2}{3} - \frac{x^2}{2} \right]_{\sqrt{y}}^{2-y} dy$$

$$= \int_0^1 \left[\frac{(2-y)^3y^2}{3} - \frac{(2-y)^2}{2} \right] - \left[\frac{y^{\frac{7}{2}}}{3} - \frac{y}{2} \right] dy$$

- The remaining straightforward integral is left to you the student to do.

$\square$

Problem 29

(b) Find an equivalent iterated integral expression for the double integral in (a), where the order of integration is reversed from the order used in part (a). (Do **not** evaluate this integral.)

Solution:

$$\int\int_{\mathbf{R}} (x^2 y - x)\, dA$$

$$= \int_0^1 \int_0^{x^2} (x^2 y - x)\, dy\, dx + \int_1^2 \int_0^{2-x} (x^2 y - x)\, dy\, dx.$$

$\square$

Problem 30

Calculate the line integral
$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r},$$
where $\mathbf{F}(x, y) = y^2 x\mathbf{i} + xy\mathbf{j}$, and $\mathbf{C}$ is the path starting at $(1, 2)$, moving along a line segment to $(3, 0)$ and then moving along a second line segment to $(0, 1)$.

Solution:

- Let $\mathbf{C}_1$ be the segment from $(1, 2)$ to $(3, 0)$ and let $\mathbf{C}_2$ be the segment from $(3, 0)$ to $(0, 1)$.
- Parameterizations for these curves are:
  $\mathbf{C}_1(t) = \langle 1 + 2t, 2 - 2t\rangle \qquad 0 \leq t \leq 1$
  $\mathbf{C}_2(t) = \langle 3 - 3t, t\rangle \qquad 0 \leq t \leq 1$
- Next calculate:

$$\int_{\mathbf{C}} y^2 x\, dx + xy\, dy = \int_{\mathbf{C}_1} y^2 x\, dx + xy\, dy + \int_{\mathbf{C}_2} y^2 x\, dx + xy\, dy =$$

$$\int_0^1 (2-2t)^2(1+2t)2\, dt + (1+2t)(2-2t)(-2)\, dt + \int_0^1 t^2(3-3t)(-3)\, dt + (3-3t)t\, dt.$$

- I leave the remaining long but straightforward calculation to you the student.

$\square$

Problem 31

Evaluate the integral $$\int\int_{\mathbf{R}} y\sqrt{x^2+y^2}\, dA$$

with $\mathbf{R}$

the region $\{(x,y)\colon 1 \le x^2+y^2 \le 2,\ \ 0 \le y \le x.\}$

Solution:

- First describe the domain $\mathbf{R}$ in polar coordinates:

$$\mathbf{R} = \{1 \le r \le \sqrt{2},\ \ 0 \le \theta \le \frac{\pi}{4}\}.$$

- Rewrite the integral in polar coordinates:

$$\int\int_{\mathbf{R}} y\sqrt{x^2+y^2}\, dA = \int_0^{\frac{\pi}{4}}\int_1^{\sqrt{2}} (r\sin\theta\, r) r\, dr\, d\theta.$$

- Now evaluating the integral:

$$\int_0^{\frac{\pi}{4}}\int_1^{\sqrt{2}} \sin\theta\, r^3 dr\, d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{1}{4}\sin\theta\, r^4\right]_1^{\sqrt{2}} d\theta$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{4}} \sin\theta\, d\theta = \frac{3}{4}\left(-\cos\theta\right)\Big|_0^{\frac{\pi}{4}} = \frac{3}{4}\left(1-\frac{1}{\sqrt{2}}\right).$$

$\square$

Problem 32

(a) Show that the vector field $\mathbf{F}(x, y) = \left\langle \frac{1}{y} + 2x, -\frac{x}{y^2} + 1 \right\rangle$ is conservative by finding a **potential function** $f(x, y)$.

Solution:

- Suppose $f(x, y)$ is the desired **potential function**.
- Then: $f_x(x, y) = \frac{1}{y} + 2x \Longrightarrow f(x, y) = \frac{x}{y} + x^2 + g(y)$, where $g(y)$ is a function of y.
- Then:

$$f_y(x, y) = \frac{-x}{y^2} + g'(y) = \frac{-x}{y^2} + 1$$

$$\Longrightarrow g'(y) = 1 \Longrightarrow g(y) = y + \mathbf{K},$$

  where $\mathbf{K}$ is a constant.
- Taking $\mathbf{K} = 0$, we obtain: $f(x, y) = \frac{x}{y} + x^2 + y$.

$\square$

Problem 32

(b) Let $\mathbf{C}$ be the path described by the parametric curve $\mathbf{r}(t) = \langle 1 + 2t, 1 + t^2 \rangle$ for (a) to determine the value of the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr}$.

Solution:

By the Fundamental Theorem of Calculus for line integrals,

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr} = f(x_2, y_2) - f(x_1, y_1),$$

where (x_1, y_1) is the beginning point for $\mathbf{r}(t)$ and (x_2, y_2) is the ending point for $\mathbf{r}(t)$. Note that the person who made this problem forgot to give the beginning and ending times, so this is all we can do.

$\square$

Problem 33

(a) Find the equation of the **tangent plane** at the point $P = (1, 1 - 1)$ in the level surface $f(x, y, z) = 3x^2 + xyz + z^2 = 1$.

Solution:

- Recall that the gradient field $\nabla f(x, y, z)$ is orthogonal to the level set surfaces of $f(x, y)$. Hence $\mathbf{n} = \nabla f(1, 1, -1)$ is a normal vector for the **tangent plane**.
- Calculating:
$$\nabla f = \langle 6x + yz, xz, xy + 2z \rangle$$
$$\nabla f(1, 1, -1) = \langle 6 - 1, -1, 1 - 2 \rangle = \langle 5, -1, -1 \rangle.$$
- The equation of the **tangent plane** is:
$$\langle 5, -1, -1 \rangle \cdot \langle x - 1, y - 1, z + 1 \rangle = 5(x - 1) - (y - 1) - (z + 1) = 0.$$

□

Problem 33

(b) Find the **directional derivative** of the function $f(x, y, z)$ at $P = (1, 1, -1)$ in the direction of the tangent vector to the space curve $\mathbf{r}(t) = \langle 2t^2 - t, t^{-2}, t^2 - 2t^3 \rangle$ at $t = 1$.

Solution:

- First find the tangent vector $\mathbf{v}$ to $\mathbf{r}(t)$ at $t = 1$:

$$\mathbf{r}'(t) = \langle 4t - 1, -2t^{-3}, 2t - 6t^2 \rangle$$
$$\mathbf{v} = \mathbf{r}'(1) = \langle 3, -2, -4 \rangle.$$

- The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 3, -2, -4 \rangle}{\sqrt{29}}.$$

- By part **(a)**, ∇f at $(1, 1, -1)$ is:

$$\nabla f(1, 1, -1) = \langle 5, -1, -1 \rangle.$$

- The **directional derivative** is:

$$D_{\mathbf{u}} f(1, 1, -1) = \nabla f(1, 1, -1) \cdot \frac{1}{\sqrt{29}} \langle 3, -2, -4 \rangle$$
$$= \frac{1}{\sqrt{29}} (\langle 5, -1, -1 \rangle \cdot \langle 3, -2, -4 \rangle = \frac{21}{\sqrt{29}}.$$

$\square$

Problem 34

Find the absolute **maxima** and **minima** of the function

$$f(x, y) = x^2 - 2xy + 2y^2 - 2y.$$

in the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 7$.

Solution:

This problem is left to you the student to do. $\square$

Problem 35

Consider the function $f(x, y) = xe^{xy}$. Let P be the point $(1, 0)$.

(a) Find the rate of change of the function f at the point P in the direction of the point $Q = (3, 2)$.

Solution:

- The unit vector $\mathbf{u}$ in the direction of the point $(3, 2)$ is:

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\langle 2, 2 \rangle}{\sqrt{4+4}} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle.$$

- Calculate $\nabla f(1, 0)$:

$$\nabla f = \langle e^{xy} + xye^{xy}, x^2 e^{xy} \rangle$$

$$\nabla f(1, 0) = \langle 1, 1 \rangle.$$

- The **directional derivative** is:

$$D_{\mathbf{u}} f(0, 1) = \nabla f(1, 0) \cdot \mathbf{u} = \langle 1, 1 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle = \frac{2}{\sqrt{2}}.$$

□

Problem 35

Consider the function $f(x, y) = xe^{xy}$. Let P be the point $(1, 0)$.

(b) Give a direction in terms of a unit vector (there are two possibilities) for which the rate of change of f at P in that direction is zero.

Solution:

- Let $\mathbf{v} = \langle a, b \rangle$ be a vector with $a^2 + b^2 = 1$ and suppose $D_{\mathbf{v}} f(1, 0) = 0$.
- Calculating, we get:

$$D_{\mathbf{v}} f(1, 0) = \nabla f(0, 1) \cdot \langle a, b \rangle = \langle 1, 1 \rangle \cdot \langle a, b \rangle = a + b.$$

- Setting $D_{\mathbf{v}} f(1, 0) = 0$ means $a + b = 0 \Longrightarrow a = -b$.
- Then $a^2 + b^2 = (-b)^2 + b^2 = 2b^2 = 1 \Longrightarrow b = \pm\frac{1}{\sqrt{2}}$.
- Hence, the two possibilities for $\mathbf{v}$ are: $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ and $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

□

Problem 36

(a) Find the work done by the vector field $\mathbf{F}(x, y) = \langle x - y, x \rangle$ over the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$.

Solution:

- The work $\mathbf{W}$ done is:
$$\mathbf{W} = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} (x - y)\, dx + x\, dy.$$
- Note $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ parameterizes $\mathbf{C}$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$.
- So, the work done is:
$$\begin{aligned}\mathbf{W} &= \int_0^{2\pi} (\cos t - \sin t)(-\sin t)\, dt + \int_0^{2\pi} \cos t \cos t\, dt \\ &= -\int_0^{2\pi} \sin(t)\cos(t)\, dt + \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt \\ &= -\left.\frac{\sin^2(t)}{2}\right|_0^{2\pi} + 2\pi = 2\pi.\end{aligned}$$

$\square$

Problem 36

(b) Use Green's Theorem to calculate the line integral $\int_{\mathbf{C}}(-y^2)\,dx + xy\,dy$, over the **positively** (counterclockwise) oriented closed curve **C** defined by $x = 1,\ y = 1$ and the coordinate axes.

Solution:

- Recall that Green's Theorem is:

$$\int_{\mathbf{C}} P\,dx + Q\,dy = \int\int_{\mathbf{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

  where **D** is the square bounded by **C**.

- Applying Green's Theorem, we obtain:

$$\int_{\mathbf{C}}(-y^2)\,dx+(xy)\,dy = \int_0^1\int_0^1 (y+2y)\,dy\,dx = 3\int_0^1\int_0^1 y\,dy\,dx$$

$$= 3\int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = 3\int_0^1 \frac{1}{2}\,dx = \frac{3}{2}.$$

□

Problem 37

(a) Show that the vector field $\mathbf{F}(x,y) = \langle x^2y, \frac{1}{3}x^3\rangle$ is conservative and find a function f such that $\mathbf{F} = \nabla f$.

Solution:

- Since
$$\frac{\partial}{\partial y}(x^2y) = x^2 = \frac{\partial}{\partial x}(\frac{1}{3}x^3),$$
the vector field is conservative.
- Suppose $f(x,y)$ is a **potential function** for $\mathbf{F}(x,y)$. Then:
$$f_x(x,y) = x^2y \Longrightarrow f(x,y) = \int x^2y\, dx = \frac{x^3y}{3} + g(y),$$
where $g(y)$ is a function of y.
- Then:
$$f_y(x,y) = \frac{\partial}{\partial y}(\frac{x^3y}{3} + g(y)) = \frac{1}{3}x^3 + g'(y) = \frac{1}{3}x^3$$
$$\Longrightarrow g'(y) = 0 \Longrightarrow g(y) = \mathbf{C},$$
for some constant $\mathbf{C}$.
- Setting $\mathbf{C} = 0$, we get: $f(x,y) = \dfrac{x^3y}{3}$. $\square$

Problem 37

(b) Using the result in part (a), calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, along the curve **C** which is the arc of $y = x^4$ from $(0,0)$ to $(2,16)$.

Solution:

- By the Fundamental Theorem of Calculus,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = f(2,16) - f(0,0) = \frac{8 \cdot 16}{3} - 0 = 128.$$

$\square$

Problem 38

Consider the surface $x^2 + y^2 - \frac{1}{4}z^2 = 0$ and the point $P(1, 2, -2\sqrt{5})$ which lies on the surface.

(a) Find the equation of the **tangent plane** to the surface at the point P.

(b) Find the equation of the **normal line** to the surface at the point P.

Solution:

- Let $\mathbf{F}(x, y, z) = x^2 + y^2 - \frac{1}{4}z^2$ and note that $\nabla\mathbf{F}$ is normal to the level set surface $x^2 + y^2 - \frac{1}{4}z^2 = 0$.
- Calculating $\nabla\mathbf{F}$ at $(1, 2, -2\sqrt{5})$, we obtain:
  $\nabla\mathbf{F} = \langle 2x, 2y, -\frac{1}{2}z\rangle \qquad \nabla\mathbf{F}(1, 2, -2\sqrt{5}) = \langle 2, 4, \sqrt{5}\rangle$.
- The equation of the **tangent plane** is:

$$\langle 2, 4, \sqrt{5}\rangle \cdot \langle x-1, y-2, z+2\sqrt{5}\rangle = 2(x-1)+4(y-2)+\sqrt{5}(z+2\sqrt{5}) = 0.$$

- The **vector equation** of the **normal line** is:
  $\mathbf{L}(t) = \langle 1, 2, -2\sqrt{5}\rangle + t\langle 2, 4, \sqrt{5}\rangle = \langle 1+2t, 2+4t, -2\sqrt{5}+\sqrt{5}t\rangle$. $\square$

Problem 39

A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate (including the boundary $x^2 + y^2 = 1$) is heated so that the temperature at any point (x, y) on the plate is given by $\mathbf{T}(x, y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^2 + y^2 = 1$.

Solution:

- First find the critical points:

$$\nabla \mathbf{T} = \langle 2x - 1, 4y \rangle = \langle 0, 0 \rangle \implies x = \frac{1}{2} \text{ and } y = 0.$$

- Apply **Lagrange multipliers** with the constraint function $g(x, y) = x^2 + y^2 = 1$:

$$\nabla \mathbf{T} = \langle 2x - 1, 4y \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle,$$

$$\implies 4y = \lambda 2y \implies \lambda = 2 \quad \text{or} \quad y = 0.$$

- $y = 0 \implies x = \pm 1.$
- $\lambda = 2 \implies 2x - 1 = 2(2x) = 4x \implies x = -\frac{1}{2}$ and $y = \pm \frac{\sqrt{3}}{2}$.
- Checking values:
  $f(\frac{1}{2}, 0) \ = -\frac{1}{4}, \ \ f(1, 0) = 0, \ \ f(-1, 0) = 2, \ \ f(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}) = \frac{9}{4}.$
- The **maximum** value is $\frac{9}{4}$ and the **minimum** value is $-\frac{1}{4}$. $\square$

Problem 40

The acceleration of a particle at any time t is given by

$$\mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2\rangle,$$

while its initial velocity is $\mathbf{v}(0) = \langle 0, 3, 0\rangle$. At what times, if any are the velocity and the acceleration of the particle orthogonal?

Solution:

- First find the velocity $\mathbf{v}(t)$ by integrating $\mathbf{a}(t)$ and using the initial value $\mathbf{v}(0) = \langle 0, 3, 0\rangle$:
$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle -3\cos t, -3\sin t, 2\rangle\, dt$$
$$= \langle -3\sin t + x_0, 3\cos t + y_0, 2t + z_0\rangle.$$
- Since $\mathbf{v}(0) = \langle 0, 3, 0\rangle$, we get
$\langle -3\sin(0) + x_0, 3\cos(0) + y_0, 2\cdot 0 + z_0\rangle = \langle x_0, 3 + y_0, z_0\rangle = \langle 0, 3, 0\rangle,$
$$\implies x_0 = y_0 = z_0 = 0.$$
- Hence, $$\mathbf{v}(t) = \langle -3\sin t, 3\cos t, 2t\rangle.$$
- Take dot products and solve for t:
$$\mathbf{a}(t)\cdot\mathbf{v}(t) = 9\cos t\sin t - 9\sin t\cos t + 4t = 4t = 0$$
$$\implies t = 0.$$

□

Problem 41(a) - Spring 2009

Consider the two lines $\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$. Find the point at which they intersect.

Solution.

- To find the point of intersection, set $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. That is,

$$1 + t = 2 - s;$$

$$1 - t = s;$$

$$2t = 2.$$

- Solve these equations to obtain $t = 1, s = 0$.
- This gives the point of intersection $(2, 0, 2)$.

$\square$

Problem 41(b) - Spring 2009

Consider the two lines $\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$. Find a normal vector of the plane which contains these two lines.

Solution.

- Note that the vectors parallel to the two lines are $\mathbf{v}_1 = \langle 1, -1, 2\rangle$ and $\mathbf{v}_2 = \langle -1, 1, 0\rangle$, respectively.
- So a normal vector of the plane which contains these two lines is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \langle -2, -2, 0\rangle$$

□

Problem 41(c) - Spring 2009

Consider the two lines $\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$. Find the cosine of the angle between the lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$.

Solution.

- The vectors parallel to the two lines are $\mathbf{v}_1 = \langle 1, -1, 2\rangle$ and $\mathbf{v}_2 = \langle -1, 1, 0\rangle$, respectively.
- Denote by θ the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{\langle 1, -1, 2\rangle \cdot \langle -1, 1, 0\rangle}{|\langle 1, -1, 2\rangle||\langle -1, 1, 0\rangle|} = \frac{-1}{\sqrt{3}}$$

- Since the angle between two lines is either acute or right, the cosine of the angle between the lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ is $\frac{1}{\sqrt{3}}$. $\square$

Problem 42(a) - Spring 2009

Consider the surface **S** defined by the equation $xy^2 + xz + z^2 = 7$. Find the equation of the **tangent plane** to the surface **S** at the point $(1, 1, 2)$.

Solution.

- Let $f(x, y, z) = xy^2 + xz + z^2$. Compute the gradient:

$$\nabla f(x, y, z) = \langle y^2 + z, 2xy, x + 2z \rangle.$$

- So a normal vector of the **tangent plane** to the surface **S** at the point $(1, 1, 2)$ is:

$$\mathbf{n} = \nabla f(1, 1, 2) = \langle 3, 2, 5 \rangle.$$

- Thus an equation of the **tangent plane** to the surface **S** at the point $(1, 1, 2)$ is:

$$3(x - 1) + 2(y - 1) + 5(z - 2) = 0.$$

□

Problem 42(b) - Spring 2009

Find the **directional derivative** of $f(x,y,z) = x^2 + xy^2 + z$ at the point $(1,2,3)$ in the direction $\mathbf{v} = \langle 1,2,2\rangle$.

Solution.

- Compute the gradient of f at $(1,2,3)$:

  $$\nabla f(x,y,z) = \langle 2x + y^2, 2xy, 1\rangle \Rightarrow \nabla f(1,2,3) = \langle 6,4,1\rangle$$

- Normalize the direction: $\mathbf{u} = \frac{\langle 1,2,2\rangle}{|\langle 1,2,2\rangle|} = \frac{1}{3}\langle 1,2,2\rangle$.
- Evaluate:
  $D_{\mathbf{u}} f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u} = \langle 6,4,1\rangle \cdot \frac{1}{3}\langle 1,2,2\rangle = \frac{16}{3}$.

$\square$

Problem 43(a) - Spring 2009

Find the volume **V** of the solid under the graph of $z = 9 - x^2 - y^2$, inside the cylinder $x^2 + y^2 = 1$, and above the xy-plane.

Solution.

- The volume $\mathbf{V} = \int\int_{\mathbf{D}} 9 - x^2 - y^2 \, dA$ where $\mathbf{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$.
- To compute this double integral, we use polar coordinates:

$$\int\int_{\mathbf{D}} 9 - x^2 - y^2 \, dA = \int_0^{2\pi} \int_0^1 (9 - r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (9r - r^3) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{17}{4} d\theta = \frac{17}{4} \theta \Big|_0^{2\pi} = \frac{17}{4}(2\pi - 0) = \frac{17}{2}\pi.$$

- So the volume $\mathbf{V} = \frac{17\pi}{2}$.

$\square$

Problem 43(b) - Spring 2009

Evaluate the iterated integral $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$.

Solution.

- The iterated integral $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$ is equal to the double integral $\int\int_{\mathbf{D}} e^{y^2}\, dA$, where

$$\mathbf{D} = \{(x, y) \,|\, 0 \leq x \leq 1,\, x \leq y \leq 1\}.$$

- $\mathbf{D}$ can be rewritten as a type II region

$$\mathbf{D} = \{(x, y) \,|\, 0 \leq y \leq 1,\, 0 \leq x \leq y\}.$$

- So,

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

$$= \int_0^1 x e^{y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 y e^{y^2}\, dy = \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{e}{2} - \frac{1}{2}. \quad \square$$

Problem 44(a) - Spring 2009

Determine whether or not the vector field $\mathbf{F}(x, y) = \langle 3e^{2y} + 2xy, xe^{2y} + x^2 + y^2 \rangle$ is **conservative**. If so, find a potential function.

Solution:

- A vector field $\mathbf{F} = \langle P, Q \rangle$ is **conservative** on a simply connected open domain if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on that domain and the component functions of the vector field are defined and have continuous partial derivatives. Since $\mathbb{R}^2$ is a simply connected open set, it is enough to check that P, Q satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
- For the vector field $\mathbf{F} = \langle 3e^{2y} + 2xy, xe^{2y} + x^2 + y^2 \rangle$, we have

$$\frac{\partial P}{\partial y} = 6e^{2y} + 2x \neq e^{2y} + 2x = \frac{\partial Q}{\partial x}$$

- So we conclude $\mathbf{F}$ is **NOT conservative**. □

Problem 44(b) - Spring 2009

Determine whether or not the vector field $\mathbf{G}(x, y) = \langle 3x^2 + 2y^2, 4xy + 3 \rangle$ is **conservative**. If so, find a potential function.

Solution:

- A vector field $\mathbf{G} = \langle P, Q \rangle$ is **conservative** on a simply connected open domain if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on that domain and the component functions of the vector field are defined and have continuous partial derivatives. Since $\mathbb{R}^2$ is simply connected open set, it is enough to check that P, Q satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
- For the vector field $\mathbf{G} = (3x^2 + 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$, we have

$$\frac{\partial P}{\partial y} = 4y = \frac{\partial Q}{\partial x}$$

- So we conclude $\mathbf{G}$ is **conservative**. $\square$

Continuation of Solution to 44(b):

- A potential function for $\mathbf{G}$ is a function g such that $\mathbf{G} = \nabla g$. To compute g we solve,
$$\mathbf{G} = \langle 3x^2 + 2y^2, 4xy + 3 \rangle = \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \rangle = \nabla g.$$
- Anti-differentiate the component equations with respect to the appropriate variable:
$$g = \int 3x^2 + 2y^2 \, dx = x^3 + 2xy^2 + h_1(y)$$
$$= \int 4xy + 3 \, dy = 2xy^2 + 3y + h_2(x)$$
- So we have $h_1(y) = 3y$ and $h_2(x) = x^3$. Thus
$$g = x^3 + 2xy^2 + 3y.$$
- Be sure to check your answer! Compute
$$\nabla g = \langle 3x^2 + 2y^2, 4xy + 3 \rangle = \mathbf{G}$$

□

Problem 45(a) - Spring 2009

Find parametric equations for the line segment **C** from the point $A = (-1, 5, 0)$ to the point $B = (1, 6, 4)$.

Solution:

- To get **parametric equations** for **L** you need a point through which the line passes and a vector parallel to the line. For example, take the point to be A and the vector to be $\overrightarrow{AB}$.
- The vector equation of **C** is

$$\mathbf{C}(t) = \overrightarrow{OA} + t\overrightarrow{AB} = \langle -1, 5, 0 \rangle + t \langle 2, 1, 4 \rangle = \langle -1 + 2t, 5 + t, 4t \rangle,$$

where O is the origin.

- So the **parametric equations** are:

$$\left| \begin{array}{l} x = -1 + 2t \\ y = 5 + t, \qquad\qquad t \in [0, 1]. \\ z = 4t \end{array} \right.$$

$\square$

Problem 45(b) - Spring 2009

Evaluate the line integral $\int_{\mathbf{C}} xz^2\,dy + y\,dz$, where **C** is the line segment in part (a).

Solution.

- By part (a), the **parametric equations** of **C** are:

$$\left|\begin{array}{l} x = -1 + 2t \\ y = 5 + t, \qquad\qquad t \in [0,1]. \\ z = 4t \end{array}\right.$$

- So,

$$\int_{\mathbf{C}} xz^2\,dy + y\,dz = \int_0^1 (-1+2t)(4t)^2\,dt + (5+t)\,4dt$$

$$= 4\int_0^1 -4t^2 + 8t^3 + 5 + t\,dt =$$

$$4\left(-\frac{4}{3}t^3 + 2t^4 + 5t + \frac{1}{2}t^2\right)\Big|_0^1 = 4\left(-\frac{4}{3} + 2 + 5 + \frac{1}{2}\right) = \frac{74}{3}.$$

□

Problem 46 - Spring 2009

Let **D** be the region on the plane bounded by the curves $y = 2x - x^2$ and the x-axis, and **C** be the positively oriented boundary curve of **D**. Use **Green's Theorem** to evaluate the line integral $\oint_{\mathbf{C}} (xy + \cos(e^x))\,dx + (x^2 + e^{\cos y})\,dy$.

Solution:

- Green's Theorem states $\oint_{\mathbf{C}} P\,dx + Q\,dy = \int\int_{\mathbf{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$
- In this problem, $P = xy + \cos(e^x)$ and $Q = x^2 + e^{\cos y}$.
- So, $\oint_{\mathbf{C}} (xy + \cos(e^x))\,dx + (x^2 + e^{\cos y})\,dy =$ $\int\int_{\mathbf{D}} \left(\frac{\partial(x^2+e^{\cos y})}{\partial x} - \frac{\partial(xy+\cos(e^x))}{\partial y}\right) dA = \int\int_{\mathbf{D}} (2x - x)dA =$ $\int_0^2 \int_0^{2x-x^2} x\,dydx = \int_0^2 xy\,\Big|_0^{2x-x^2} dx = \int_0^2 x(2x - x^2)\,dx =$ $\int_0^2 2x^2 - x^3\,dx = \frac{2}{3}x^3 - \frac{1}{4}x^4\,\Big|_0^2 = \frac{2}{3}(8) - \frac{1}{4}(16) = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}.$

$\square$

Solutions to the remaining problems are similar to worked out solutions here and in the other reviews and are left to you the students to do on your own.

47. Find **parametric equations** for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

48. Find the equation of the plane containing the points $P(1, 3, 0)$, $Q(2, -1, 2)$ and $R(0, 0, 1)$.

49. Find all points of intersection of the parametric curve $\mathbf{r}(t) = \langle 2t^2 - 2, t, 1 - t - t^2 \rangle$ and the plane $x + y + z = 3$.

50. Find the absolute **maximum** and **minimum** of the function $f(x, y) = x^2 + 2y^2 - 2y$ on the closed disc $x^2 + y^2 \leq 5$ of radius $\sqrt{5}$.

51. Evaluate $\int\int_{\mathbf{R}} xy \, dA$, where **R** is the region in the first quadrant bounded by the line $y = 2x$ and the parabola $y = x^2$.

52. Consider the vector field $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1 \rangle$.
    (a) Show that $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1 \rangle$ is conservative by finding a **potential function** $f(x, y)$ for $\mathbf{F}(x, y)$.
    (b) Use your answer to part **a** to evaluate the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr}$, where **C** is the arc of the parabola $y = x^2$ going from $(0, 0)$ to $(2, 4)$.

53. Evaluate the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr}$ where $\mathbf{F} = \langle y^2 + \sin x, xy \rangle$ and **C** is the unit circle oriented counterclockwise.

54. Evaluate the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = \langle y^2, 2xy + x \rangle$ and **C** is the curve starting at $(0, 0)$, traveling along a line segment to $(2, 1)$ and then traveling along a second line segment to $(0, 3)$.