Hi students!

I am putting this old version of my review for the second midterm review, place and time to be announced. Check for updates on the web site as to which sections of the book will actually be covered. **Enjoy!!**
Best, **Bill Meeks**

PS. There are probably errors in some of the solutions presented here and for a few problems you need to complete them or simplify the answers; some questions are left to you the student. Also you might need to add more detailed explanations or justifications on the actual similar problems on your exam. I will keep updating these solutions with better corrected/improved versions. The first 5 slides are from Exam 1 practice problems but the material might fall on our Exam 2.

Problem 26(a) - Exam 1 - Fall 2006

Let $g(x,y) = ye^x$. Estimate $g(0.1, 1.9)$ using the **linear approximation** $\mathbf{L}(x,y)$ of $g(x,y)$ at $(x,y) = (0,2)$.

Solution:

- Calculating **partial derivatives** at $(0,2)$, we obtain:
$$g_x(x,y) = ye^x \qquad g_y(x,y) = e^x$$
$$g_x(0,2) = 2 \qquad g_y(0,2) = 1.$$

- Let $\mathbf{L}(x,y)$ be the **linear approximation** at $(0,2)$.
$$\mathbf{L}(x,y) = g(0,2) + g_x(0,2)(x-0) + g_y(0,2)(y-2)$$
$$\mathbf{L}(x,y) = 2 + 2x + (y-2).$$

- Calculating at $(0.1, 1.9)$:
$$\mathbf{L}(0.1, 1.9) = 2 + 2(0.1) + (1.9 - 2) = 2 + .2 - .1 = 2.1$$

□

Problem 36 - Exam 1

Find an equation for the **tangent plane** to the graph of $f(x, y) = y \ln x$ at $(1, 4, 0)$.

Solution:

- Recall that the **tangent plane** to a surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Calculating **partial derivatives**, we obtain:

$$f_x(x, y) = \frac{y}{x} \qquad f_y(x, y) = \ln x$$

$$f_x(1, 4) = 4 \qquad f_y(1, 4) = \ln 1 = 0.$$

- The equation of the **tangent plane** is:

$$z = 4(x - 1) + 0 \cdot (y - 4) = 4x - 4.$$

□

Problem 40 - Exam 1

Explain why the **limit** of $f(x, y) = (3x^2y^2)/(2x^4 + y^4)$ does not exist as (x, y) approaches $(0, 0)$.

Solution:

- Along the line $\langle t, t\rangle$, $t \neq 0$, $f(x, y)$ has the constant value $\frac{3}{3} = 1$.
- Along the line $\langle 0, t\rangle$, $t \neq 0$, $f(x, y)$ has the constant value $\frac{0}{1} = 0$.
- Since $f(x, y)$ has **2 different limiting values** at $(0, 0)$, it does **not** have a **limit** at $(0, 0)$.

$\square$

Problem 42(a) - Exam 1

Find all of the first order and second order **partial derivatives** of the function $f(x, y) = x^3 - xy^2 + y$.

Solution:

- First calculate the first order **partial derivatives**:

$$f_x(x, y) = 3x^2 - y^2 \qquad f_y(x, y) = -2xy + 1.$$

- The second order **partial derivatives** f_{xx}, f_{xy}, f_{yx} and f_{yy} are:

$$f_{xx}(x, y) = 6x \qquad f_{xy}(x, y) = -2y$$
$$f_{yx}(x, y) = -2y \qquad f_{yy}(x, y) = -2x.$$

$\square$

Problem 43 - Exam 1

Find the **linear approximation** $\mathbf{L}(x, y)$ of the function $f(x, y) = xye^x$ at $(x, y) = (1, 1)$, and use it to estimate $f(1.1, 0.9)$.

Solution:

- Calculating **partial derivatives** at $(1, 1)$, we obtain:

$$f_x(x, y) = ye^x + xye^x \qquad f_y(x, y) = xe^x$$

$$f_x(1, 1) = 2e \qquad f_y(1, 1) = e.$$

- Let $\mathbf{L}(x, y)$ be the **linear approximation** at $(1, 1)$.

$$\mathbf{L}(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

$$\mathbf{L}(x, y) = e + 2e(x - 1) + e(y - 1).$$

- Calculating at $(1.1, 0.9)$, we obtain:

$$\mathbf{L}(1.1, 0.9) = e + 2e(0.1) + e(-0.1) = 1.1e.$$

$\square$

Problem 1 - Exam 2 - Fall 2008

1. For the function $f(x,y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}:
   - $f_x(x,y) = 4x + y^2$
   - $f_y(x,y) = 2xy$
   - $f_{xy}(x,y) = 2y$
   - $f_{xx}(x,y) = 4$
2. What is the **gradient** $\nabla f(x,y)$ of f at the point $(1,2)$?
   $\nabla f = \langle f_x, f_y \rangle = \langle 4x + y^2, 2xy \rangle$ $\qquad \nabla f(1,2) = \langle 8, 4 \rangle$.
3. Calculate the **directional derivative** of f at the point $(1,2)$ in the direction of the vector $\mathbf{v} = \langle 3, 4 \rangle$?
   - $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{5}\langle 3, 4 \rangle$ is the unit vector in the direction of $\langle 3, 4 \rangle$.
   - Next evaluate
     $D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle 8, 4 \rangle \cdot \frac{1}{5}\langle 3, 4 \rangle = \frac{1}{5}(24 + 16) = 8.$
4. What is the **linearization** $\mathbf{L}(x,y)$ of f at $(1,2)$?
   $$\mathbf{L}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$
   $$= 6 + 8(x-1) + 4(x-2).$$
5. Use the **linearization** $\mathbf{L}(x,y)$ in the previous part to estimate $f(0.9, 2.1)$.
   $\mathbf{L}(0.9, 2.1) = 6 + 8(0.9 - 1) + 4(2.1 - 2) = 6 - .8 + .4 = 5.6$

Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. Suppose the hiker is now at the point $P(2, -1, 79)$ heading North, is she **ascending** or **descending**? Justify your answers.

Solution:

- Let $f(x, y) = z = 100 - 4x^2 - 5y^2$.
- This is a problem where we need to calculate the sign of the **directional derivative** $D_{\langle 0,1\rangle} f(2, -1) = \nabla f(2, -1) \cdot \langle 0, 1\rangle$, where $\langle 0, 1\rangle$ represents **North**.
- Calculating, we obtain:

$$\nabla f(x, y) = \langle -8x, -10y\rangle \qquad \nabla f(2, -1) = \langle -16, 10\rangle.$$

- Hence,

$$D_{\langle 0,1\rangle} f(2, -1) = \langle -16, 10\rangle \cdot \langle 0, 1\rangle = 10 > 0,$$

  which means that she is **ascending**.

$\square$

Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. Justify your answers.

When the hiker is at the point $Q(1, 0, 96)$, in which direction on the map should she initially head to **descend** most rapidly?

Solution:

- Recall that $\nabla f(x, y) = \langle -8x, -10y \rangle$.
- The direction of **greatest descent** is in the direction $\mathbf{v}$ of $-\nabla f$ at the point $(1, 0)$ in the xy-plane.
- Thus,

$$\mathbf{v} = -\nabla f(1, 0) = \langle 8, 0 \rangle,$$

which means that she should go **East**. $\square$

Problem 2(c) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. What is her **rate** of **descent** when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1, 0, 96)$? Justify your answers.

Solution:

- By velocity, we mean the velocity of the projection on the xy-plane or map (the wording is somewhat ambiguous).
- By part (b), if she travels at unit speed (in measurements on the map which we don't know) in the direction of $\nabla f(1, 0)$ (which is **East** $= \langle 1, 0 \rangle$), then her maximal rate of decent is $|\nabla f(1, 0)| = 8$ (in measurements of the map).
- So, her rate of decent in the direction of greatest decent is

$$10 \cdot 8 \text{ meters/minute} = 80 \text{ meters/minute.}$$

$\square$

Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. When the hiker is at the point $Q(1, 0, 96)$, in which two directions on her map can she initially head to **neither** ascend nor descend (to keep traveling at the same height)? Justify your answers.

Solution:

- First find all the possible vectors $\mathbf{v} = \langle x, y \rangle$ which are **orthogonal** to $\nabla f(1, 0) = \langle -8, 0 \rangle$:

$$\langle -8, 0 \rangle \cdot \langle x, y \rangle = -8x + 0 = 0 \Longrightarrow x = 0.$$

- Therefore, at the point $Q(1, 0, 96)$ and in the map directions of the vectors $\pm\langle 0, 1 \rangle$, she is neither ascending or descending. These directions are **North** and **South**.

$\square$

Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the **partial derivatives** $f_x(x, y)$ and $f_y(x, y)$ at certain points (x, y)

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	-2	4
-1	2	3	-1
1	2	-1	3

(You are given more values than you will need for this problem.) Suppose that x and y are functions of variable t: $x = t^3$; $y = t^2 + 1$, so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$.

Solution:

- By the **Chain Rule** we have:

$$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x} \cdot 3t^2 + \frac{\partial f}{\partial y} \cdot 2t.$$

- Note that when $t = 1$, then $x = 1$ and $y = 2$ and that $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t \Longrightarrow \frac{dx}{dt}(1) = 3$ and $\frac{dy}{dt}(1) = 2$.
- Plug in the values in the table into the **Chain Rule** at $t = 1$:

$$f'(1) = \frac{\partial f}{\partial x}(1,2) \cdot 3 + \frac{\partial f}{\partial y}(1,2) \cdot 2 = (-1) \cdot 3 + 3 \cdot 2 = 3.$$

$\square$

Problem 3(b) - Fall 2008

Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v^2, \; y = u - v^2.$$

Solution:

- By the **Chain Rule** we have:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= (3x^2y^2 + y^3)(2v) + (2x^3y + 3y^2x)(-2v).$$

- When $u = 1$ and $v = 1$, then $x = 1^2 + 1^2 = 2$ and $y = 1 - 1^2 = 0$.
- So for $u = 1$ and $v = 1$, we get:

$$\frac{\partial z}{\partial v} = 0.$$

□

Problem 4 - Fall 2008

Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1, 1, 1)$ which lies on the surface.

(i) Find the equation of the **tangent plane** to the surface at P.

(ii) Find the equation of the **normal line** to the surface at P.

Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z) = x^2 + y^2 - 2z^2$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$\nabla \mathbf{F}(x, y, z) = \langle 2x, 2y, -4z \rangle$$
$$\mathbf{n} = \nabla \mathbf{F}(1, 1, 1) = \langle 2, 2, -4 \rangle.$$

- The equation of the **tangent plane** is:

$$\langle 2, 2, -4 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 2(x-1)+2(y-1)-4(z-1) = 0.$$

- The **vector equation** of the **normal line** is:

$$\mathbf{L}(t) = \langle 1, 1, 1 \rangle + t\langle 2, 2, -4 \rangle = \langle 1 + 2t, 1 + 2t, 1 - 4t \rangle.$$

$\square$

Problem 5 - Fall 2008

Let
$$f(x,y) = 2x^3 + xy^2 + 6x^2 + y^2.$$
Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

Solution:

- First calculate $\nabla f(x,y)$ and set to $\langle 0,0\rangle$:
$$\nabla f(x,y) = \langle 6x^2 + y^2 + 12x, 2xy + 2y\rangle = \langle 0,0\rangle.$$
- This gives the following two equations:
$$6x^2 + y^2 + 12x = 0$$
$$2xy + 2y = y(2x+2) = 0 \Longrightarrow y = 0 \;\text{ or }\; x = -1.$$
- If $x = -1$, then the first equation gives:
$$6 + y^2 - 12 = y^2 - 6 = 0 \Longrightarrow y = \sqrt{6} \;\text{ or }\; y = -\sqrt{6}.$$
- If $y = 0$, then the first equation gives $x = 0$ or $x = -2$.
- The set of **critical points** is:
$$\{(0,0), (-2,0), (-1,\sqrt{6}), (-1,-\sqrt{6})\}.$$

□

Problem 5 - Fall 2008

Let
$$f(x,y) = 2x^3 + xy^2 + 6x^2 + y^2.$$
Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

Solution: Continuation of problem 5.

- Recall that $\{(0,0), (-2,0), (-1,\sqrt{6}), (-1,-\sqrt{6})\}$ is the set of **critical points**.
- Since we will apply the **Second Derivative Test**, we first write down the second derivative matrix:

$$\mathbf{D} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x+12 & 2y \\ 2y & 2x+2 \end{vmatrix}$$

- Since $D(0,0) = 12 \cdot 2 = 24 > 0$ and $f_{xx}(0) = 12 > 0$, then $(0,0)$ is a local **minimum**.
- Since $D(-2,0) = 24 > 0$ and $f_{xx}(-2,0) = -12 < 0$, then $(-2,0)$ is a local **maximum**.
- Since $D(-1,\sqrt{6}) < 0$, then $(-1,\sqrt{6})$ is a **saddle** point.
- Since $D(-1,-\sqrt{6}) < 0$, then $(-1,-\sqrt{6})$ is a **saddle** point. $\square$

Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^2 + y^2 \leq 4$. The plate (including the boundary $x^2 + y^2 = 4$) is heated so that the temperature at any point (x, y) on the plate is given by $\mathbf{T}(x, y) = x^2 + y^2 - 2x$. Find the temperatures at the **hottest** and the coldest points on the plate, including the boundary $x^2 + y^2 = 4$.

Solution:

- We first find the critical points.
$$\nabla\mathbf{T} = \langle 2x - 2, 2y\rangle = 0 \Longrightarrow x = 1 \text{ and } y = 0.$$
- Next use **Lagrange Multipliers** to study max and min of f on the boundary circle $g(x, y) = x^2 + y^2 = 4$:
$\nabla\mathbf{T} = \langle 2x - 2, 2y\rangle = \lambda\nabla g = \lambda\langle 2x, 2y\rangle$.
- $2y = \lambda 2y \Longrightarrow y = 0$ or $\lambda = 1$.
- $y = 0 \Longrightarrow x = \pm 2$.
- $\lambda = 1 \Longrightarrow 2x - 2 = 2x$, which is impossible.
- Now check the value of $\mathbf{T}$ at 3 points:
$$\mathbf{T}(1, 0) = -1, \quad \mathbf{T}(2, 0) = 0, \quad \mathbf{T}(-2, 0) = 8.$$
- **Maximum** temperature is 8 and the **minimum** temperature is -1.

□

Problem 7(a) - Spring 2008

Consider the equation $x^2 + y^2/9 + z^2/4 = 1$.
Identify this quadric (i.e. quadratic surface), and graph the portion of the surface in the region $x \geq 0, y \geq 0$, and $z \geq 0$. Your graph should include tick marks along the three positive coordinate axes, and must clearly show where the surface intersects any of the three positive coordinate axes.

Solution:

This is an **ellipsoid**. A problem of this type will **not** be on this midterm. □

Problem 7(b) - Spring 2008

Consider the equation $x^2 + y^2/9 + z^2/4 = 1$. Calculate z_x and z_y at an arbitrary point (x, y, z) on the surface (wherever possible).

Solution:

- Recall the following formulas of **implicit differentiation** of $\mathbf{F}(x, y, z) = x^2 + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0$:

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \qquad \frac{\partial z}{\partial y} = \frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}}.$$

- Plugging in the following values,

$$\frac{\partial \mathbf{F}}{\partial x} = 2x \qquad \frac{\partial \mathbf{F}}{\partial y} = \frac{2}{9}y \qquad \frac{\partial \mathbf{F}}{\partial z} = \frac{1}{2}z,$$

yields

$$z_x = \frac{-2x}{\frac{1}{2}z} = -4 \cdot \frac{x}{z} \qquad z_y = \frac{-\frac{2}{9}y}{\frac{1}{2}z} = -\frac{4}{9} \cdot \frac{y}{z},$$

which make sense when $z \neq 0$. □

Problem 7(c) - Spring 2008

Consider the equation $x^2 + y^2/9 + z^2/4 = 1$.
Determine the equation of the **tangent plane** to the surface at the point $(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1)$.

Solution:

- For $\mathbf{F}(x, y, z) = x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n} = \nabla\mathbf{F}(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1)$:

$$\nabla\mathbf{F}(x, y, z) = \langle 2x, \frac{2}{9}y, \frac{1}{2}z\rangle$$

$$\mathbf{n} = \nabla\mathbf{F}(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1) = \langle \frac{2}{\sqrt{2}}, \frac{6}{18}, \frac{1}{2}\rangle = \langle \sqrt{2}, \frac{1}{3}, \frac{1}{2}\rangle.$$

- The equation of the **tangent plane** is:

$$\begin{aligned} 0 &= \nabla\mathbf{F}(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1) \cdot \langle x - \frac{1}{\sqrt{2}}, y - \frac{3}{2}, z - 1\rangle \\ &= \langle \sqrt{2}, \frac{1}{3}, \frac{1}{2}\rangle \cdot \langle x - \frac{1}{\sqrt{2}}, y - \frac{3}{2}, z - 1\rangle \\ &= \sqrt{2}(x - \frac{1}{\sqrt{2}}) + \frac{1}{3}(y - \frac{3}{2}) + \frac{1}{2}(z - 1) = 0. \end{aligned}$$

□

Problem 8(a) - Spring 2008

Given the function $f(x,y) = x^2y + ye^{xy}$.
Find the **linearization** $\mathbf{L}(x,y)$ of f at the point $(0,5)$ and use it to approximate the value of f at the point $(.1, 4.9)$. (An unsupported numerical approximation to $f(.1, 4.9)$ will not receive credit.)

Solution:

- Calculating **partial derivatives** at $(0,5)$, we obtain:

$$f_x(x,y) = 2xy + y^2e^{xy} \qquad f_y(x,y) = x^2 + e^{xy} + xye^xy$$

$$f_x(0,5) = 25 \qquad f_y(0,5) = 1.$$

- Let $\mathbf{L}(x,y)$ be the **linear approximation** at $(0,5)$.

$$\mathbf{L}(x,y) = f(0,5) + f_x(0,5)(x-0) + f_y(0,5)(y-5)$$

$$\mathbf{L}(x,y) = 5 + 25x + (y-5).$$

- Calculating at $(.1, 4.9)$:

$$\mathbf{L}(.1, 4.9) = 5 + 25(0.1) + (4.9-5) = 7.4.$$

$\square$

Problem 8(b) - Spring 2008

Given the function $f(x, y) = x^2 y + y e^{xy}$.
Suppose that $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. Calculate f_θ at $r = 5$ and $\theta = \frac{\pi}{2}$.

Solution:

- The **Chain Rule** gives

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= (2xy + y^2 e^{xy})(-r\sin\theta) + (x^2 + e^{xy} + xye^{xy})(r\cos\theta).$$

- When $r = 5$ and $\theta = \frac{\pi}{2}$, then $x = 0$ and $y = 5$.
- Thus,

$$\frac{\partial f}{\partial \theta} = (0 + 25e^0)(-5) + 0 = -125.$$

$\square$

Problem 8(c) - Spring 2008

Given the function $f(x, y) = x^2y + ye^{xy}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t) = f(x(t), y(t))$ where $f(x, y)$ is the function defined above. Calculate $\mathbf{F}'(3)$, assuming that at time $t = 3$ the particle's position is $(x(3), y(3)) = (0, 5)$ and its velocity is $(x'(3), y'(3)) = (3, -2)$.

Solution:

- The **Chain Rule** gives

$$\frac{d\mathbf{F}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2xy + y^2e^{xy})\frac{dx}{dt} + (x^2 + e^{xy} + xye^{xy})\frac{dy}{dt}.$$

- Plugging in values, we obtain:

$$\mathbf{F}'(3) = (0 + 5^2e^0)(3) + (0 + e^0 + 0)(-2) = 75 + (-2) = 73.$$

$\square$

Problem 9(a) - Spring 2008

Consider the function $f(x, y) = 2\sqrt{x^2 + 4y}$.
Find the **directional derivative** of $f(x, y)$ at $P = (-2, 3)$ in the direction starting from P pointing towards $Q = (0, 4)$.

Solution:

- First calculate **partial derivatives** of $f(x, y) = 2(x^2 + 4y)^{\frac{1}{2}}$:

$$f_x = \frac{2x}{\sqrt{x^2 + 4y}} \qquad f_y = \frac{4}{\sqrt{x^2 + 4y}}.$$

- So,

$$\nabla f(-2, 3) = \langle \frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}} \rangle = \langle -1, 1 \rangle.$$

- The unit vector $\mathbf{u}$ in the direction $\overrightarrow{PQ} = \langle 2, 1 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{5}} \langle 2, 1 \rangle$.
- $D_{\mathbf{u}} f(-2, 3) = \nabla f(-2, 3) \cdot \mathbf{u} = \langle -1, 1 \rangle \cdot \frac{1}{\sqrt{5}} \langle 2, 1 \rangle = -\frac{1}{\sqrt{5}}$. $\square$

Problem 9(b) - Spring 2008

Consider the function $f(x, y) = 2\sqrt{x^2 + 4y}$.
Find all unit vectors $\mathbf{u}$ for which the **directional derivative** $D_{\mathbf{u}}f(-2, 3) = 0$.

Solution:

- First find all the possible non-unit vectors $\mathbf{v} = \langle x, y\rangle$ which are **orthogonal** to $\nabla f(-2, 3) = \langle -1, 1\rangle$:

$$\langle -1, 1\rangle \cdot \langle x, y\rangle = -x + y = 0 \Longrightarrow x = y.$$

- Therefore, $\mathbf{v} = \langle x, x\rangle$ works for any $x \neq 0$.
- The set of unit vectors $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ such that $D_{\mathbf{u}}f(-2, 3) = \nabla f(-2, 3) \cdot \mathbf{u} = 0$ consists of 2 vectors:

$$\{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle, \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle\}.$$

$\square$

Problem 9(c) - Spring 2008

Consider the function $f(x, y) = 2\sqrt{x^2 + 4y}$.
Is there a unit vector $\mathbf{u}$ for which the **directional derivative** $D_{\mathbf{u}}f(-2, 3) = 4$? Either find the appropriate $\mathbf{u}$ or explain why not.

Solution:

- First recall that:

$$\nabla f = \langle \frac{2x}{\sqrt{x^2 + 4y}}, \frac{4}{\sqrt{x^2 + 4y}} \rangle \qquad \nabla f(-2, 3) = \langle -1, 1 \rangle.$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u} = \langle x, y \rangle$ such that

$$D_{\mathbf{u}}f(-2, 3) = \nabla f(-2, 3) \cdot \mathbf{u} = \langle -1, 1 \rangle \cdot \mathbf{u} = 4.$$

- If such $\mathbf{u}$ exists, then

$$4 = |\langle -1, 1 \rangle \cdot \mathbf{u}| = |\langle -1, 1 \rangle| \cdot |\mathbf{u}||\cos\theta| = \sqrt{2}\,|\cos\theta| \leq \sqrt{2}.$$

- Therefore, **no** such unit vector $\mathbf{u}$ exists.

$\square$

Problem 10(a) - Spring 2008

Let $f(x, y) = \frac{2}{3}x^3 + \frac{1}{3}y^3 - xy$.
Find all **critical points** of $f(x, y)$.

Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0, 0\rangle$:

$$\nabla f(x, y) = \langle 2x^2 - y, y^2 - x\rangle = \langle 0, 0\rangle$$

- The first coordinate equation $2x^2 - y = 0$ implies $y = 2x^2$.
- Plugging $y = 2x^2$ into the second coordinate equation gives $4x^4 - x = x(4x^3 - 1) = 0 \Longrightarrow x = 0$ or $x = 4^{-\frac{1}{3}}$.
- Hence, $(x = 0 \text{ and } y = 0)$ or $(x = 4^{-\frac{1}{3}} \text{ and } y = 2 \cdot 4^{-\frac{2}{3}})$.
- This gives a set of two **critical points**:

$$\{(0, 0), (4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}})\}.$$

□

Problem 10(b) - Spring 2008

Let $f(x,y) = \frac{2}{3}x^3 + \frac{1}{3}y^3 - xy$.
Classify each **critical point** as a relative **maximum**, relative (local) **minimum** or **saddle**; you do not need to calculate the function at these points, but your answer must be justified.

Solution:

- By part **(a)** $\nabla f = \langle 2x^2 - y, y^2 - x \rangle$ and the set of critical points is $\{(0,0), (4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}})\}$.
- Now write down the Hessian:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4x & -1 \\ -1 & 2y \end{vmatrix} = 8xy - 1.$$

- Next apply the **Second Derivative Test**.
- Since $D(0,0) = -1 < 0$, then $(0,0)$ is a **saddle** point.
- Since $D(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}) = 4 - 1 > 0$ and $f_{xx}(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}) = 4 \cdot 4^{-\frac{1}{3}} = 4^{\frac{2}{3}} > 0$, then $(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}})$ is a local **minimum**. □

Problem 11 - Spring 2008

Use the method of **Lagrange multipliers** to determine all points (x, y) where the function $f(x, y) = 2x^2 + 4y^2 + 16$ has an **extreme value** (either a **maximum** or a **minimum**) subject to the **constraint** $\frac{1}{4}x^2 + y^2 = 4$.

Solution:

- Set $g(x, y) = \frac{1}{4}x^2 + y^2$.
- Set $\nabla f = \langle 4x, 8y \rangle = \lambda \nabla g = \lambda \langle \frac{1}{2}x, 2y \rangle$ and solve:

$$8y = 2\lambda y \Longrightarrow \lambda = 4 \text{ or } y = 0.$$

$$4x = \frac{1}{2}\lambda x \Longrightarrow \lambda = 8 \text{ or } x = 0.$$

- Since λ cannot simultaneously be 4 and 8, then x or y is zero.
- From the **constraint** $\frac{1}{4}x^2 + y^2 = 4$, $x = 0 \Longrightarrow y = \pm 2$ and $y = 0 \Longrightarrow x = \pm 4$.
- We need to check the values of f at the points $(0, \pm 2)$, $(\pm 4, 0)$:

$$f(0, \pm 2) = 32 \qquad\qquad f(\pm 4, 0) = 48.$$

- Hence, $f(x, y)$ has its **minimum** value of 32 at the points $(0, \pm 2)$ and its **maximum** value of 48 at the points $(\pm 4, 0)$. $\square$

Problem 12 - Fall 2007

Find the x and y coordinates of all **critical points** of the function

$$f(x,y) = 2x^3 - 6x^2 + xy^2 + y^2$$

and use the **Second Derivative Test** to classify them as local **minima**, local **maxima** or **saddle** points.

Solution:

- First calculate $\nabla f(x,y)$ and set equal to $(0,0)$:
$$\nabla f(x,y) = \langle 6x^2 - 12x + y^2, 2xy + 2y \rangle = \langle 0,0 \rangle.$$
$\Longrightarrow 2y(x+1) = 0$ and so $y = 0$ or $x = -1$.
- Suppose $y = 0$. Then $6x^2 - 12x = 6x(x-2) = 0$ and $x = 0$ or $x = 2$.
- Suppose $x = -1$. Then $6 + 12 + y^2 = 0$, which is impossible.
- The set of **critical points** is $\{(0,0),(2,0)\}$.
- Next calculate:
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x - 12 & 2y \\ 2y & 2x+2 \end{vmatrix} = 24x^2 - 24 - 4y^2.$$
- $D(0,0) = -24 < 0$ and so $(0,0)$ is a **saddle** point.
- $D(2,0) = 96 - 24 = 72 > 0$ and $f_{xx}(2,0) = 12 > 0$, so $(2,0)$ is a local **minimum**. □

Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 1 - 4x^2 - 3y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. Suppose the hiker is now at the point $P(\frac{1}{4}, -\frac{1}{2}, 0)$ heading North, is she **ascending** or **descending**? Justify your answers.

Solution:

- Let $f(x, y) = z = 1 - 4x^2 - 3y^2$.
- This is a problem where we need to calculate the sign of the **directional derivative** $D_{\langle 0,1\rangle} f(\frac{1}{4}, -\frac{1}{2}) = \nabla f(\frac{1}{4}, -\frac{1}{2}) \cdot \langle 0, 1\rangle$, where $\langle 0, 1\rangle$ represents **North**.
- Calculating, we obtain:

$$\nabla f(x, y) = \langle -8x, -6y\rangle \qquad \nabla f(\frac{1}{4}, -\frac{1}{2}) = \langle -2, 3\rangle.$$

- Hence,

$$D_{\langle 0,1\rangle} f(\frac{1}{4}, -\frac{1}{2}) = \langle -2, 3\rangle \cdot \langle 0, 1\rangle = 3 > 0,$$

which means that she is **ascending**. $\square$

Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 1 - 4x^2 - 3y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**. Justify your answers.

When the hiker is at the point $Q(\frac{1}{4}, 0, \frac{3}{4})$, in which direction should she initially head to **ascend** most rapidly?

Solution:

- Recall that $\nabla f(x, y) = \langle -8x, -6y \rangle$.
- The direction of **greatest ascent** is in the direction $\mathbf{v} = \nabla f$ at the point $(\frac{1}{4}, 0)$ in the xy-plane.
- Thus,

$$\mathbf{v} = \nabla f(\frac{1}{4}, 0) = \langle -2, 0 \rangle$$

which means that she should go **West**. □

Problem 14 - Fall 2007

Find the **volume** **V** of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0, y = 3$ and $z = 0$.

Solution:

- Note that the graph of $f(x, y) = z = 6 - xy$ is nonnegative over the rectangle **R** $= [-2, 2] \times [0, 3]$ and the **volume** **V** described is the **volume** under the graph.
- Applying Fubini's Theorem gives:

$$\mathbf{V} = \int_{-2}^{2} \int_{0}^{3} 6 - xy \, dy \, dx = \int_{-2}^{2} \left[6y - \frac{1}{2} x y^2 \right]_0^3 dx$$

$$= \int_{-2}^{2} (18 - \frac{9}{2} x) dx = \left. 18x - \frac{9}{4} x^2 \right|_{-2}^{2} = (36 - 9) - (-36 - 9) = 72.$$

$\square$

Problem 15 - Fall 2007

Let $z(x, y) = x^2 + y^2 - xy$ where $x = s - r$ and $y = y(r, s)$ is an **unknown function** of r and s. (Note that z can be considered a function of r and s.) Suppose we **know** that

$$y(2, 3) = 3, \quad \frac{\partial y}{\partial r}(2, 3) = 7, \quad \text{and } \frac{\partial y}{\partial s}(2, 3) = -5.$$

Calculate $\frac{\partial z}{\partial r}$ when $r = 2$ and $s = 3$.

Solution:

- By the **Chain Rule**:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x - y)\frac{\partial x}{\partial r} + (2y - x)\frac{\partial y}{\partial r}.$$

- Note that $r = 2$ and $s = 3 \Longrightarrow x = 1$ and $y = 3$.
- Hence,

$$\frac{\partial z}{\partial r} = (2 - 3)(-1) + (6 - 1)7 = 1 + 35 = 36.$$

$\square$

Problem 16(a) - Fall 2007

Let $\mathbf{F}(x, y, z) = x^2 - 2xy - y^2 + 8x + 4y - z$.
Write the equation of the **tangent plane** to the surface given by $\mathbf{F}(x, y, z) = 0$ at the point $(-2, 1, -5)$.

Solution:

- Note that the normal $\mathbf{n}$ of the plane is $\nabla\mathbf{F}(-2, 1, -5)$.
- Calculating, we obtain:

$$\nabla\mathbf{F}(x, y, z) = \langle 2x - 2y + 8, -2x - 2y + 4, -1\rangle.$$

- So,

$$\mathbf{n} = \nabla\mathbf{F}(-2, 1, -5) = \langle -4 - 2 + 8, 4 - 2 + 4, -1\rangle = \langle 2, 6, -1\rangle.$$

- The equation of the **tangent plane** is:

$$\mathbf{n} \cdot \langle x + 2, y - 1, z + 5\rangle = 2(x + 2) + 6(y - 1) - (z + 5) = 0.$$

$\square$

Problem 16(b) - Fall 2007

Find the point (a, b, c) on the surface $\mathbf{F}(x, y, z) = 0$ at which the **tangent plane** is **horizontal**, that is, parallel to the $z = 0$ plane.

Solution:

- Since $\nabla\mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z) = 0$, a **horizontal tangent plane** to the surface occurs where $\nabla\mathbf{F}$ is vertical.
- $\nabla\mathbf{F}$ is vertical on $\mathbf{F}(x, y, z) = 0$, when its first 2 coordinates vanish:
  $$\nabla\mathbf{F} = \langle 2x - 2y + 8, -2x - 2y + 4, -1\rangle = \langle 0, 0, -1\rangle \implies$$
  $$\begin{aligned} 2x - 2y + 8 &= 0 \\ -2x - 2y + 4 &= 0 \end{aligned}$$
- Adding these equations $\implies 4y = 12 \implies y = 3$.
- Plugging in $y = 3$ in first equation gives
  $$2x - 2 \cdot 3 + 8 = 0 \implies x = -1.$$
- $\mathbf{F}(x, y, z) = x^2 - 2xy - y^2 + 8x + 4y - z$ and $\mathbf{F}(-1, 3, z) = 0$,
  $\implies z = (-1)^2 - 2(-1)(3) - 3^2 + 8(-1) + 4(3) = 2.$
- The unique point with **horizontal tangent plane** is $(-1, 3, 2)$.

$\square$

Problem 17 - Fall 2007

Find the points on the ellipse $x^2 + 4y^2 = 4$ that are closest to the point $(1, 0)$.

Solution:

- We approach this problem using the method of **Lagrange multipliers**. Let $f(x, y)$ be square of the distance function from $(1, 0)$ to an arbitrary point (x, y) in $\mathbb{R}^2$.
- We must find the **minimum** of $f(x, y) = (x - 1)^2 + y^2$, subject to the **constraint** $g(x, y) = x^2 + 4y^2 = 4$ (distance squared to $(1, 0)$).
- Calculating for some $\lambda \in \mathbb{R}$,
  $\nabla f(x, y) = \langle 2(x - 1), 2y \rangle = \lambda \nabla g(x, y) = \lambda \langle 2x, 8y \rangle$.
- Hence, $2y = \lambda 8y \implies \lambda = \frac{1}{4}$ or $y = 0$.
- If $y = 0$, then the **constraint** implies $x = \pm 2$.
- If $\lambda = \frac{1}{4}$, then $2(x - 1) = \lambda 2x = \frac{1}{2}x \implies x = \frac{4}{3}$.
- If $x = \frac{4}{3}$, then the **constraint** implies $y = \pm \frac{\sqrt{5}}{3}$.
- The function $f(x, y)$ has its **minimum** value at one of the 4 points $(\pm 2, 0)$ and $(\frac{4}{3}, \pm \frac{\sqrt{5}}{3})$, and one easily checks its **minimum** value of $\frac{2}{3}$ occurs at the 2 points $(\frac{4}{3}, \pm \frac{\sqrt{5}}{3})$. □

Problem 18(a) - Fall 2006

Let $f(x,y)$ be a differentiable function with the following values of the **partial derivatives** $f_x(x,y)$ and $f_y(x,y)$ at certain points (x,y)

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	-2	4
-1	2	3	-1
1	2	-1	1

(You are given more values than you will need for this problem.) Suppose that x and y are functions of variable t: $x = t^3$; $y = t^2 + 1$, so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$.

Solution:

- By the **Chain Rule** we have:

$$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x} \cdot 3t^2 + \frac{\partial f}{\partial y} \cdot 2t.$$

- Note that when $t = 1$, then $x = 1$ and $y = 2$ and that $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t \implies \frac{dx}{dt}(1) = 3$ and $\frac{dy}{dt}(1) = 2$.
- Plug in the values in the table into the **Chain Rule** at $t = 1$:

$$f'(1) = \frac{\partial f}{\partial x}(1,2) \cdot 3 + \frac{\partial f}{\partial y}(1,2) \cdot 2 = (-1) \cdot 3 + 1 \cdot 2 = -1. \quad \square$$

Problem 18(b) - Fall 2006

Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v, y = 2u - v.$$

Solution:

- When $u = 1$ and $v = 1$, then $x = 1^2 + 1 = 2$, $y = 2 \cdot 1 - 1 = 1$, $\frac{\partial x}{\partial v} = 1$ and $\frac{\partial y}{\partial v} = -1$.
- By the **Chain Rule** we have:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= (3x^2y^2 + y^3)(1) + (2x^3y + 3y^2x)(-1) = 3x^2y^2 + y^3 - 2x^3y - 3y^2x.$$

- So for $u = 1$ and $v = 1$, we get:

$$\frac{\partial z}{\partial v}(1,1) = 3 \cdot 4 + 1 - 2 \cdot 8 - 3 \cdot 2 = -9.$$

$\square$

Problem 19(a) - Fall 2006

Let $f(x, y) = x^2y^3 + y^4$. Find the **directional derivative** of f at the point $(1, 1)$ in the direction which forms an angle (counterclockwise) of $\pi/6$ with the positive x-axis.

Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$\mathbf{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle.$$

- Calculating the **gradient**, we get:

$$\nabla f(x, y) = \langle 2xy^3, 3x^2y^2 + 4y^3 \rangle;$$

$$\nabla f(1, 1) = \langle 2, 3 + 4 \rangle = \langle 2, 7 \rangle.$$

- So the **directional derivative** is:

$$D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = \langle 2, 7 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \sqrt{3} + \frac{7}{2}.$$

□

Problem 19(b) - Fall 2006

Find an equation of the **tangent line** to the curve $x^2y + y^3 - 5 = 0$ at the point $(x, y) = (2, 1)$.

Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y) = x^2y + y^3 - 5 = 0$ at the point $(2, 1)$ is $\nabla\mathbf{F}(2, 1)$.
- Calculating, we obtain:

$$\nabla\mathbf{F}(x, y) = \langle 2xy, x^2 + 3y^2\rangle;$$

$$\mathbf{n} = \nabla\mathbf{F}(2, 1) = \langle 4, 7\rangle.$$

- The equation of the **tangent line** is:

$$\mathbf{n}\cdot\langle x-2, y-1\rangle = \langle 4, 7\rangle\cdot\langle x-2, y-1\rangle = 4(x-2)+7(y-1) = 0.$$

□

Problem 20 - Fall 2006

Let
$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$
Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

Solution:

- First calculate $\nabla f(x,y)$ and set to $\langle 0,0 \rangle$:
$$\nabla f(x,y) = \langle 6x^2 + y^2 + 10x, 2xy + 2y \rangle = \langle 0,0 \rangle.$$
- This gives the following two equations:
$$6x^2 + y^2 + 10x = 0$$
$$2xy + 2y = y(2x+2) = 0 \Longrightarrow y = 0 \;\text{ or }\; x = -1.$$
- If $x = -1$, then the first equation gives:
$$6 + y^2 - 10 = y^2 - 4 = 0 \Longrightarrow y = 2 \;\text{ or }\; y = -2.$$
- If $y = 0$, then the first equation gives $x = 0$ or $x = -\frac{5}{3}$.
- The set of **critical points** is:
$$\{(0,0), (-\frac{5}{3}, 0), (-1,2), (-1,-2)\}.$$

$\square$

Problem 20 - Fall 2006

Let
$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$
Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

Solution: Continuation of problem 20.

- Recall that $\{(0,0), (-\frac{5}{3},0), (-1,2), (-1,-2)\}$ is the set of **critical points**.
- Since we will apply the **Second Derivative Test**, we first write down the second derivative matrix:
$$\mathbf{D} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{vmatrix}$$
- Since $D(0,0) = 10 \cdot 2 = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, then $(0,0)$ is a local **minimum**.
- Since $D(\frac{-5}{3},0) = -10 \cdot (-\frac{4}{3}) > 0$ and $f_{xx}(\frac{-5}{3},0) = -10 < 0$, then $(-\frac{5}{3},0)$ is a local **maximum**.
- Since $D(-1,2) < 0$, then $(-1,2)$ is a **saddle** point.
- Since $D(-1,-2) < 0$, then $(-1,-2)$ is a **saddle** point. $\square$

Problem 21 - Fall 2006

Find the **maximum** value of $f(x, y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 1$ (Hint: Use **Lagrange Multipliers**).

Solution:

- The **constraint** function is $g(x, y) = x^2 + y^2$. Note that x and y cannot both be 0.
- Set $\nabla f = \langle 4x, 2y \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$ and solve:

$$4x = \lambda 2x \implies x = 0 \ \text{ or } \ \lambda = 2.$$

$$2y = \lambda 2y \implies y = 0 \ \text{ or } \ \lambda = 1.$$

- Since λ cannot simultaneously be 2 and 1, then x or y is zero.
- From the **constraint** $x^2 + y^2 = 1$, $x = 0 \implies y = \pm 1$ and $y = 0 \implies x = \pm 1$.
- We only need to check the values of f at the points $(0, \pm 1)$, $(\pm 1, 0)$:

$$f(0, \pm 1) = 1 \qquad\qquad f(\pm 1, 0) = 2.$$

- $f(x, y)$ has its **maximum** value 2 at the points $(\pm 1, 0)$. $\square$

Problem 22 - Fall 2006

Find the **volume V** above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z = 5 + x^2 + y$.

Solution:

We apply Fubini's Theorem:

$$\mathbf{V} = \int_2^5 \int_{-1}^1 (5 + x^2 + y)\, dx\, dy = \int_2^5 \left[5x + \frac{x^3}{3} + yx \right]_{-1}^1 dy$$

$$= \int_2^5 10 + \frac{2}{3} + 2y \; dy = (10 + \frac{2}{3})y + y^2 \Big|_2^5 = 53.$$

$\square$

Problem 23 - Fall 2006

Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$$

by reversing the order of integration.

Solution:

There is no integration problem on this exam with varying limits of integration (function limits). $\square$

Problem 24(1)

Use **Chain Rule** to find dz/dt.
$z = x^2y + 2y^3$, $x = 1 + t^2$, $y = (1-t)^2$.

Solution:

- Calculating:
$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -2(1-t),$$
$$\frac{\partial z}{\partial x} = 2xy \qquad \frac{\partial z}{\partial y} = x^2 + 6y^2.$$
- By the **Chain Rule**,
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2xy \cdot 2t + (x^2+6y^2)\cdot(-2(1-t))$$
$$= 2(1+t^2)(1-t)^2 2t + ((1+t^2)^2 + 6(1-t)^4)(-2(1-t)).$$
- You can simplify further if you want.

$\square$

Problem 24(2)

Use **Chain Rule** to find $\partial z/\partial u$ and $\partial z/\partial v$.
$z = x^3 + xy^2 + y^3$, $x = uv$, $y = u + v$.

Solution:

- Calculating:

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$
$$\frac{\partial y}{\partial u} = 1 \qquad \frac{\partial y}{\partial v} = 1$$
$$\frac{\partial z}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial z}{\partial y} = 2xy + 3y^2.$$

- By the **Chain Rule**:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$= (3x^2+y^2)v+(2xy+3y^2) = (3u^2v^2+(u+v)^2)v+2uv(u+v)+3(u+v)^2,$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= (3x^2+y^2)u+(2xy+3y^2) = (3u^2v^2+(u+v)^2)u+(2uv(u+v)+3(u+v)^2).$$

$\square$

Problem 25

If $z = f(x, y)$, where f is differentiable, and $x = 1 + t^2$, $y = 3t$, compute dz/dt at $t = 2$ provided that $f_x(5, 6) = f_y(5, 6) = -1$.

Solution:

- We apply the **Chain Rule** $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
- Since $t = 2$, then $x(2) = 1 + 2^2 = 5$ and $y(2) = 3 \cdot 2 = 6$.
- Calculating, we obtain:

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3.$$

- Evaluate using the **Chain Rule**:

$$\frac{dz}{dt}(2) = \frac{\partial f}{\partial x}(5,6)\frac{dx}{dt}(2) + \frac{\partial f}{\partial y}(5,6)\frac{dy}{dt}(2)$$

$$= -1(2 \cdot 2) + (-1)3 = -7.$$

□

Problem 26(a)

For the functions

1. $f(x, y) = x^2y + y^3 - y^2$,
2. $g(x, y) = x/y + xy$,
3. $h(x, y) = \sin(x^2y) + xy^2$

find the **gradient** at $(0, 1)$.

Solution:

1. $\nabla f(x, y) = \langle 2xy, x^2 + 3y^2 - 2y \rangle$
   $\nabla f(0, 1) = \langle 0, 1 \rangle$;
2. $\nabla g(x, y) = \langle \frac{1}{y} + y, -\frac{x}{y^2} + x \rangle$
   $\nabla g(0, 1) = \langle 2, 0 \rangle$;
3. $\nabla h(x, y) = \langle \cos(x^2y)(2xy) + y^2, \cos(x^2y)x^2 + 2xy \rangle$
   $\nabla h(0, 1) = \langle 1, 0 \rangle$.

$\square$

Problem 26(b)

For the functions

1. $f(x, y) = x^2y + y^3 - y^2$,
2. $g(x, y) = x/y + xy$,
3. $h(x, y) = \sin(x^2y) + xy^2$

find the **directional derivative** at the point $(0, 1)$ in the direction of $\mathbf{v} = \langle 3, 4\rangle$.

Solution:

1. The unit vector $\mathbf{u}$ in the direction of $\mathbf{v} = \langle 3, 4\rangle$ is:
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{5}\langle 3, 4\rangle = \langle \frac{3}{5}, \frac{4}{5}\rangle.$$
2. $D_{\mathbf{u}}f(0, 1) = \nabla f(0, 1) \cdot \mathbf{u} = \langle 0, 1\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{4}{5}$.
3. $D_{\mathbf{u}}g(0, 1) = \nabla g(0, 1) \cdot \mathbf{u} = \langle 2, 0\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{6}{5}$.
4. $D_{\mathbf{u}}h(0, 1) = \nabla h(0, 1) \cdot \mathbf{u} = \langle 1, 0\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{3}{5}$.

$\square$

Problem 26(c)

For the functions

1. $f(x, y) = x^2 y + y^3 - y^2$,
2. $g(x, y) = x/y + xy$,
3. $h(x, y) = \sin(x^2 y) + xy^2$

find the **maximum rate of change** (**MRC**) at the point $(0, 1)$.

Solution:

We know that the **maximum rate of change** is the length of the **gradient** of the respective function:

$$\textbf{MRC}(f) = |\nabla f(0,1)| = |\langle 0, 1\rangle| = 1;$$

$$\textbf{MRC}(g) = |\nabla g(0,1)| = |\langle 2, 0\rangle| = 2;$$

$$\textbf{MRC}(h) = |\nabla h(0,1)| = |\langle 1, 0\rangle| = 1.$$

$\square$

Problem 27

Find an equation of the **tangent plane** to the surface $x^2 + 2y^2 - z^2 = 5$ at the point $(2, 1, 1)$.

Solution:

- For $\mathbf{F}(x, y, z) = x^2 + 2y^2 - z^2$, the **gradient** is:

$$\nabla \mathbf{F}(x, y, z) = \langle 2x, 4y, -2z \rangle.$$

- At the point $(2, 1, 1)$, we have $\nabla \mathbf{F}(2, 1, 1) = \langle 4, 4, -2 \rangle$, which is the normal vector $\mathbf{n}$ to the **tangent plane** to the surface $\mathbf{F}(x, y, z) = x^2 + 2y^2 - z^2 = 5$ at $(2, 1, 1)$.
- Since $(2, 1, 1)$ is a point on the **tangent plane**, the equation is:

$$\langle 4, 4, -2 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 4(x-2) + 4(y-1) - 2(z-1) = 0.$$

$\square$

Problem 28

Find **parametric equations** for the **tangent line** to the curve of intersection of the surfaces $z^2 = x^2 + y^2$ and $x^2 + 2y^2 + z^2 = 66$ at the point $(3, 4, 5)$.

Solution:

- If $\mathbf{n}_1$, $\mathbf{n}_2$ are the normal vectors of the respective surfaces, the equation of the **tangent line** is $\mathbf{L}(t) = \langle 3, 4, 5\rangle + t(\mathbf{n}_1 \times \mathbf{n}_2)$.
- The normal $\mathbf{n}_1$ to the surface $\mathbf{F}(x, y, z) = z^2 - x^2 - y^2 = 0$ at the point $(3, 4, 5)$ is: $\mathbf{n}_1 = \nabla\mathbf{F}(3, 4, 5) = \langle -6, -8, 10\rangle$.
- The normal $\mathbf{n}_2$ to the surface $\mathbf{G}(x, y, z) = x^2 + 2y^2 + z^2 = 66$ at the point $(3, 4, 5)$ is: $\mathbf{n}_2 = \nabla\mathbf{G}(3, 4, 5) = \langle 6, 16, 10\rangle$.
- The vector part of the line is:

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -8 & 10 \\ 6 & 16 & 10 \end{vmatrix} = \langle -240, 120, -48\rangle.$$

- The **parametric equations** are:

$$x = 3 - 240t$$
$$y = 4 + 120t$$
$$z = 5 - 48t.$$

□

Problem 29(1)

Find and classify all **critical points** (as local **maxima**, local **minima**, or **saddle** points) of the function
$f(x,y) = x^2y^2 + x^2 - 2y^3 + 3y^2$

Solution:

- Set $\nabla f = \langle 0,0\rangle$ and solve:
  $\nabla f = \langle 2xy^2 + 2x, 2x^2y - 6y^2 + 6y\rangle = \langle 0,0\rangle \Longrightarrow$
  $2xy + 2x = 2x(y^2+1) = 0 \Longrightarrow x = 0; \Longrightarrow$
  $-6y^2 + 6y = 6y(-y+1) = 0 \Longrightarrow y = 0$ or $y = 1$.
- The **critical points** are $(0,0), (0,1)$.
- The Hessian is:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y^2+2 & 4xy \\ 4xy & 2x^2 - 12y + 6 \end{vmatrix}.$$

- Since $D(0,0) = 2\cdot 6 > 0$ and $f_{xx}(0,0) = 2 > 0$, the point $(0,0)$ is a local **minimum**.
- Since $D(0,1) = 4\cdot(-12) < 0$, the point $(0,1)$ is **saddle** point. $\square$

Problem 29(2)

Find and classify all **critical points** (as local **maxima**, local **minima**, or **saddle** points) of the function $g(x, y) = x^3 + y^2 + 2xy - 4x - 3y + 5$.

Solution:

- Set $\nabla g = \langle 0, 0\rangle$ and solve:
$$\nabla g = \langle 3x^2 + 2y - 4, 2y + 2x - 3\rangle = 0$$
$$2y + 2x - 3 = 0 \Longrightarrow y = \frac{3}{2} - x.$$
- So,
$$3x^2 + 2(\frac{3}{2} - x) - 4 = 3x^2 - 2x - 1 = (3x + 1)(x - 1) = 0$$
$$\Longrightarrow x = 1 \text{ or } x = -\frac{1}{3}.$$
- The set of **critical points** is $\{(1, \frac{1}{2}), (-\frac{1}{3}, \frac{11}{6})\}$.
- The Hessian is:
$$D = \begin{vmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 2 \\ 2 & 2 \end{vmatrix}.$$
- Since $D(1, \frac{1}{2}) = (6 \cdot 2 - 4) > 0$ and $g_{xx}(1, \frac{1}{2}) = 6 > 0$, the point $(1, \frac{1}{2})$ is a local **minimum**.
- Since $D(-\frac{1}{3}, \frac{11}{6}) = -8 < 0$, then $(-1, \frac{5}{2})$ is a **saddle** point. $\square$

Problem 30

Find the **minimum** value of $f(x,y) = 3 + xy - x - 2y$ on the closed triangular region with vertices $(0,0)$, $(2,0)$ and $(0,3)$.

Solution:

- Set $\nabla f = \langle 0,0\rangle$ and solve:
  $\nabla f = \langle y-1, x-2\rangle = \langle 0,0\rangle \Longrightarrow y = 1$ and $x = 2$.
- There is exactly one **critical point** which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$,
  $f(x,0) = 3 + x\cdot 0 - x - 2\cdot 0 = 3 - x,$ which has a **minimum** value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$,
  $f(0,y) = 3 + 0\cdot y - 0 - 2y = 3 - 2y$, which has a **minimum** value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3)$, $y = -\frac{3}{2}x + 3$ and
  $f(x, y = -\frac{3}{2}x + 3) = 3 + x(-\frac{3}{2}x + 3) - x - 2(-\frac{3}{2}x + 3) = -\frac{3}{2}x^2 + 5x - 3$, which has a **minimum** of $\frac{25}{6} - 3$ at $(\frac{5}{3}, \frac{1}{2})$.
- Hence, the absolute **minimum** value of $f(x,y)$ is -3.

$\square$

Problem 31(1)

Use **Lagrange multipliers** to find the **extreme values** of $f(x,y) = xy$ with **constraint** $g(x,y) = x^2 + 2y^2 = 3$.

Solution:

- Set $\nabla f = \langle y, x\rangle = \lambda \nabla g = \lambda\langle 2x, 4y\rangle$ and solve:

$$y = \lambda 2x \implies x = 0 \text{ or } \lambda = \frac{y}{2x}.$$

$$x = \lambda 4y \implies y = 0 \text{ or } \lambda = \frac{x}{4y}.$$

- Since $g(x,y) = x^2 + 2y^2 = 3$, **either** x or y must be nonzero; the above equations then imply **both** x and y are nonzero.
- Since x, y are both nonzero, then

$$\frac{y}{2x} = \frac{x}{4y} \implies 4y^2 = 2x^2 \implies x^2 = 2y^2.$$

- From the **constraint** $x^2 + 2y^2 = 3$, we get $y = \pm\frac{\sqrt{3}}{2}$, and the 4 possible points $(\pm\frac{\sqrt{3}}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2})$ where $f(x,y)$ is **extreme**.
- Then $f(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}) = f(-\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}) = \frac{3}{2\sqrt{2}}$,
  $f(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}) = f(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}) = -\frac{3}{2\sqrt{2}}$.
- Hence, the **extreme values** are $\pm\frac{3}{2\sqrt{2}}$. $\square$

Problem 31(2)

Use **Lagrange multipliers** to find the **extreme values** of $g(x, y, z) = x + 3y - 2z$ with **constraint** $x^2 + 2y^2 + z^2 = 5$.

Solution:

There is **no Lagrange multipliers** problem in 3 variables on this exam. $\square$

Problem 32(1)

Find the iterated integral,

$$\int_1^4 \int_0^2 (x + \sqrt{y}) dx\, dy.$$

Solution:

$$\int_1^4 \int_0^2 (x + \sqrt{y}) dx\, dy = \int_1^4 \left[\frac{x^2}{2} + \sqrt{y} x \right]_0^2 dy$$

$$= \int_1^4 (2 + 2y^{\frac{1}{2}}) dy = 2y + \frac{4}{3} y^{\frac{3}{2}} \Big|_1^4$$

$$= 8 + \frac{4}{3}(8) - (2 + \frac{4}{3}).$$

$\square$

Problem 32(2)

Find the iterated integral,

$$\int_1^2 \int_0^1 (2x + 3y)^2 dy\, dx.$$

Solution:

$$\int_1^2 \int_0^1 (2x+3y)^2 \; dy\, dx = \int_1^2 \int_0^1 4x^2 + 12xy + 9y^2 \; dy\, dx$$

$$= \int_1^2 \left[4x^2y + 6xy^2 + 3y^3 \right]_0^1 \; dx = \int_1^2 4x^2 + 6x + 3 \; dx$$

$$= \frac{4}{3}x^3 + 3x^2 + 3x \Big|_1^2 = \frac{4}{3}(2)^3 + 3(2)^2 + 3(2) - (\frac{4}{3} + 3 + 3).$$

$\square$

Problem 32(3)

Find the iterated integral,

$$\int_0^1 \int_x^{2-x} (x^2 - y)dy\ dx.$$

Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

□

Problem 32(4)

Find the iterated integral,

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)dy\ dx.$$

(Hint: Reverse the order of integration)

Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

□

Problem 33(1)

Evaluate the following double integral.

$$\int\int_{\mathbf{R}} \cos(x+2y)dA, \quad \mathbf{R} = \{(x,y) \mid 0 \le x \le \pi,\, 0 \le y \le \pi/2\}.$$

Solution:

- Applying Fubini's Theorem and the fact $\sin(\pi+\theta) = -\sin(\theta)$, we obtain:

$$\int\int_{\mathbf{R}} \cos(x+2y)\,dA = \int_0^{\frac{\pi}{2}} \int_0^{\pi} \cos(x+2y)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}} \Big[\sin(x+2y)\Big]_0^{\pi}\,dy = \int_0^{\frac{\pi}{2}} \sin(\pi+2y) - \sin(2y)\,dy$$

$$= \int_0^{\frac{\pi}{2}} -\sin(2y)\,2dy = \cos(2y)\Big|_0^{\frac{\pi}{2}} = -1-1 = -2.$$

$\square$

Problem 33(2)

Evaluate the following double integral.

$$\int\int_{\mathbf{R}} e^{y^2}\, dA, \quad \mathbf{R} = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}.$$

Solution:

There is no integration problem on this exam with varying limits of integration (function limits). $\square$

Problem 33(3)

Evaluate the following double integral.

$$\int\int_{\mathbf{R}} x\sqrt{y^2 - x^2}\, dA, \quad \mathbf{R} = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}.$$

Solution:

There is no integration problem on this exam with varying limits of integration (function limits). $\square$

Problem 34(1)

Find the **volume V** of the solid under the surface $z = 4 + x^2 - y^2$ and above the rectangle

$$\mathbf{R} = \{(x, y) \mid -1 \leq x \leq 1,\, 0 \leq y \leq 2\}.$$

Solution:

$$\mathbf{V} = \int\int_{\mathbf{R}} (4 + x^2 - y^2)\, dA = \int_0^2 \int_{-1}^1 (4 + x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[4x + \frac{x^3}{3} - y^2 x\right]_{-1}^1 dy = \int_0^2 (8 + \frac{2}{3} - 2y^2)\, dy$$

$$= (8 + \frac{2}{3})y - \frac{2}{3}y^3\Big|_0^2 = \frac{52}{3} - \frac{16}{3} = \frac{36}{3}.$$

$\square$

Problem 34(2)

Find the **volume V** of the solid under the surface $z = 2x + y^2$ and above the region bounded by curves $x - y^2 = 0$ and $x - y^3 = 0$.

Solution:

There is no integration problem on this exam with varying limits of integration (function limits). $\square$

Problem 35(a) - Spring 2009

Let $f(x, y) = x^2y - y^2 - 2y - x^2$.
Find all of the critical points of f and classify them as either **local maximum**, **local minimum**, or **saddle points**.

Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$\nabla f(x, y) = \langle 2xy - 2x, x^2 - 2y - 2\rangle = \langle 0, 0\rangle$$

- The first equation $2xy - 2x = 2x(y - 1) = 0$ implies $x = 0$ or $y = 1$
- If $x = 0$, the second equation $-2y - 2 = 0 \Rightarrow y = -1$.
- If $y = 1$, the second equation $x^2 - 4 = 0 \Rightarrow x = \pm 2$.
- This gives a set of three critical points:

$$\{(0, -1), (-2, 1), (2, 1)\}.$$

□

Problem 35(a) - Spring 2009

Let $f(x,y) = x^2y - y^2 - 2y - x^2$.
Find all of the critical points of f and classify them as either **local maximum**, **local minimum**, or **saddle points**.

Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1), (-2,1), (2,1)\}$.
- Now write the Hessian of $f(x,y)$:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y-2 & 2x \\ 2x & -2 \end{vmatrix} = -4y + 4 - 4x^2$$

- Apply the Second Derivative Test.
- $D(0,-1) = 8 > 0$ and $f_{xx} = -4 < 0$, so $(0,-1)$ is a **local maximum**.
- $D(-2,1) = -16 < 0$, so $(-2,1)$ is a **saddle point**.
- $D(2,1) = -16 < 0$, so $(2,1)$ is a **saddle point**. $\square$

Problem 35(b) - Spring 2009

Let $f(x,y) = x^2y - y^2 - 2y - x^2$.
Find the **linearization** $\mathbf{L}(x,y)$ of f at the point $(1,2)$ and use it to approximate $f(0.9, 2.1)$.

Solution:

- Calculate the partial derivatives of f at $(1,2)$:

$$\nabla f(x,y) = \langle 2xy - 2x, x^2 - 2y - 2 \rangle \qquad \nabla f(1,2) = \langle 2, -5 \rangle$$

- Compute the **linearization** $\mathbf{L}(x,y)$ of f at $(1,2)$:

$$\begin{aligned} \mathbf{L}(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= -7 + 2(x-1) - 5(y-2) \end{aligned}$$

- Approximate $f(0.9, 2.1)$ by $\mathbf{L}(0.9, 2.1)$:

$$\mathbf{L}(0.9, 2.1) = -7 + 2(-0.1) - 5(0.1) = -7.7$$

$\square$

Problem 36 (a-c) - Spring 2009

Consider the function $f(x, y) = x^2 - 2xy + 3y + y^2$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y) = \langle 2x - 2y, -2x + 3 + 2y \rangle$.

(b)

Find the **directional derivative** of f at the point $(1, 0)$ in the direction $\langle 3, 4 \rangle$.

- Normalize the direction: $\mathbf{u} = \frac{\langle 3,4 \rangle}{|\langle 3,4 \rangle|} = \frac{1}{5}\langle 3, 4 \rangle$
- Evaluate: $D_u f(1, 0) = \nabla f(1, 0) \cdot \mathbf{u} = \langle 2, 1 \rangle \cdot \frac{1}{5}\langle 3, 4 \rangle = 2$.

(c)

Compute all second partial derivatives of f.

- $f_{xx}(x, y) = \frac{\partial}{\partial x}(2x - 2y) = 2$
- $f_{xy}(x, y) = f_{yx}(x, y) = \frac{\partial}{\partial y}(2x - 2y) = -2$
- $f_{yy}(x, y) = \frac{\partial}{\partial y}(-2x + 3 + 2y) = 2$.

Problem 36(d) - Spring 2009

Consider the function $f(x,y) = x^2 - 2xy + 3y + y^2$.
Suppose $x = st^2$ and $y = e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s = 2$ and $t = 1$.

Solution:

- If $s = 2$ and $t = 1$, then $x = 2 \cdot 1^2 = 2$ and $y = e^{2-1} = e$.
- The **Chain Rule** states that

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

- So,
$$\frac{\partial f}{\partial s} = (2x - 2y)(t^2) + (-2x + 3 + 2y)(e^{s-t})$$
$$= (4 - 2e) + (-4 + 3 + 2e)(e) = 4 - 3e + 2e^2.$$

-
$$\frac{\partial f}{\partial t} = (2x - 2y)(2st) + (-2x + 3 + 2y)(-e^{s-t})$$
$$= (4 - 2e)(4) + (-4 + 3 + 2e)(-e) = 16 - 7e - 2e^2.$$

□

Problem 37(a) - Spring 2009

Consider the function $f(x, y) = e^{xy}$ over the region **D** given by $x^2 + 4y^2 \leq 2$. Find the critical points of f.

Solution:

- $\nabla f(x, y) = \langle ye^{xy}, xe^{xy} \rangle = \langle 0, 0 \rangle$
- Since e^{xy} is positive, the only critical point is $(0, 0)$.

$\square$

Problem 37(b) - Spring 2009

Find the extreme values on the boundary of **D**.

Solution:

- Use Lagrange Multipliers to study the behavior of f on the boundary $g(x, y) = x^2 + 4y^2 = 2$.

$$\nabla f(x, y) = \langle ye^{xy}, xe^{xy} \rangle = \lambda \nabla g(x, y) = \lambda \langle 2x, 8y \rangle \quad (1)$$

- Since $g(0, 0) \neq 2$, $x = 0 \Rightarrow y \neq 0$, and $y = 0 \Rightarrow x \neq 0$.
- Since e^{xy} is positive and not both x and y are 0, the neither is 0 by equation 1. Hence, $\lambda / e^{xy} = y/2x = x/8y \Rightarrow 8y^2 = 2x^2$.
- Substituting into $g(x, y) = 2$ gives $2x^2 = 2$ and $8y^2 = 2$.
- There are four possible extremum points:

$$\{(-1, -\frac{1}{2}), (-1, \frac{1}{2}), (1, -\frac{1}{2}), (1, \frac{1}{2})\}$$

- So the extreme values of f on the boundary of **D** are:
  Max $= f(1, 1/2) = f(-1, -1/2) = \sqrt{e}$,
  Min $= f(1, -1/2) = f(-1, 1/2) = \frac{1}{\sqrt{e}}$.

$\square$

Problem 37(c) - Spring 2009

Consider the function $f(x, y) = e^{xy}$ over the region **D** given by $x^2 + 4y^2 \leq 2$. What is the absolute **maximum value** and absolute **minimum value** of $f(x, y)$ on **D**?

Solution:

- Recall that the only critical point of f is $(0, 0)$, and that on the boundary $\{(-1, -1/2), (-1, 1/2), (1, -1/2), (1, 1/2)\}$ are possible extremum points.
- Calculate the value of f at each point.
- $f(0, 0) = 1$
- $f(1, 1/2) = f(-1, -1/2) = \sqrt{e}$
- $f(1, -1/2) = f(-1, 1/2) = \frac{1}{\sqrt{e}}$
- So, the **maximum value** is $\sqrt{e}$ and the **minimum value** is $\frac{1}{\sqrt{e}}$.

□

Problem 38(a) - Spring 2009

Evaluate the following iterated integral.

$$\int_{-1}^{2}\int_{0}^{1}(x^2y - xy)\,dy\,dx$$

Solution:

$$\int_{-1}^{2}\int_{0}^{1}(x^2y - xy)\,dy\,dx = \int_{-1}^{2}\left[x^2\frac{y^2}{2} - x\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_{-1}^{2}\left(\frac{x^2}{2} - \frac{x}{2}\right)\,dx = \left[\frac{x^3}{6} - \frac{x^2}{4}\right]_{-1}^{2}$$

$$= \left(\frac{2^3}{6} - \frac{2^2}{4}\right) - \left(\frac{(-1)^3}{6} - \frac{(-1)^2}{4}\right) = \frac{16-12+2+3}{12} = \frac{3}{4}$$

$\square$

Problem 38(b) - Spring 2009

Find the volume **V** of the region below $z = x^2 - 2xy + 3$ and above the rectangle **R** $= [0,1] \times [-1,1]$.

Solution: Calculate using Fubini's Theorem.

$$\mathbf{V} = \int\int_{\mathbf{R}} (x^2 - 2xy + 3)\, dA = \int_0^1 \int_{-1}^1 (x^2 - 2xy + 3)\, dy\, dx$$

$$= \int_{-1}^1 \int_0^1 (x^2 - 2xy + 3)\, dx\, dy = \int_{-1}^1 \left[\frac{x^3}{3} - x^2 y + 3x\right]_0^1 dy$$

$$= \int_{-1}^1 \left(\frac{1}{3} - y + 3\right) dy = \left[\frac{y}{3} - \frac{y^2}{2} + 3y\right]_{-1}^1$$

$$= \left(\frac{1}{3} - \frac{1^2}{2} + 3(1)\right) - \left(\frac{-1}{3} - \frac{(-1)^2}{2} + 3(-1)\right) = 6 + \frac{2}{3}. \quad \square$$

Problem 39(a) - Spring 2009

Consider the surface **S** given by the equation $x^2 + y^3 + z^2 = 0$. Give an equation for the **tangent plane** of **S** at the point $(2, -2, 2)$.

Solution:

- Let $f(x, y, z) = x^2 + y^3 + z^2$.
- Compute the **gradient** of f at $(2, -2, 2)$.

$$\nabla f(x, y, z) = \langle 2x, 3y^2, 2z \rangle \qquad \nabla f(2, -2, 2) = \langle 4, 12, 4 \rangle.$$

- The equation for the **tangent plane** is

$$\nabla f(2, -2, 2) \cdot \langle x - 2, y + 2, z - 2 \rangle = 0$$

$$4(x - 2) + 12(y + 2) + 4(z - 2) = 0$$

□

Problem 39(b) - Spring 2009

Consider the surface **S** given by the equation $x^2 + y^3 + z^2 = 0$. Give an equation for the **normal line** to **S** at the point $(2, -2, 2)$.

Solution:

- Let $f(x, y, z) = x^2 + y^3 + z^2$.
- Compute the **gradient** of f at $(2, -2, 2)$.

$$\nabla f(x, y, z) = \langle 2x, 3y^2, 2z \rangle \qquad \nabla f(2, -2, 2) = \langle 4, 12, 4 \rangle.$$

- The equation for the **normal line** is

$$r(t) = \langle 2, -2, 2 \rangle + t\langle 4, 12, 4 \rangle.$$

$\square$