NOTES FOR MATH 132

Antiderivatives

Definition: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I $(\forall x \in I)$.

Recall that antiderivatives are not unique, and the following theorem gives the connection between all antiderivatives of a function.

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant $(\forall C \in \mathbf{R})$.

This most general antiderivative is also called an indefinite integral of f and is denoted by $\int f(x)dx$.

Areas

To define the area under a continuous curve $y = f(x)$ over an interval $[a, b]$ the following steps are needed:

1. divide the interval $[a, b]$ into n equal (having equal width) subintervals$[x_{i-1}, x_i]$, $i = 1, 2, ..., n$ with endpoints $x_0, x_1, ..., x_n$

$$a = x_0 < x_1 < ... < x_n = b.$$

2. constract approximating rectangles having as bases the intervals $[x_{i-1}, x_i]$, and as heights $f(x_i)$.

3. improve the approximation by increasing n, and eventually passing to the limit $n \to \infty$.

Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x].$$

Note: The limit in the definition is guaranted to exist provided the function is continious.

Definite Integral

f is a continious function on the interval $[a, b]$:

1. divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$ with endpoints $x_0, x_1, ..., x_n$:

$$a = x_0 < x_1 < ... < x_n = b.$$

2. chose sample points $x_1^*, x_2^*, ..., x_n^*$ in these subintervals ($x_i^* \in [x_{i-1}, x_i]$).
3. $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called Riemann sum for the function f over interval $[a, b]$.

Definition:The definite integral of f from a to b is defined to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

in this notation a and b are called *imits of integration* (a*-lower limit*, b*-upper limit*; f is called *integrand.*

Note: The limit in the definition is guaranted to exist provided the function is continious.

Notice that with the definition of *Definite Integral* the area under the curve $y = f(x)$ over the interval $[a, b]$ is exactly $\int_a^b f(x)dx$.

Properties of Definite Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
2. $\int_a^a f(x)dx = 0.$
3. $\int_a^b cdx = c(b-a)$ $\quad$ c is any constatnt ($\forall c \in \mathbf{R}$).
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ $\quad$ c is any constatnt ($\forall c \in \mathbf{R}$).
6. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

Comparison properties

7. if $f(x) \geq 0$ for $\forall x \in [a, b] \Rightarrow \int_a^b f(x)dx \geq 0$.
8. if $f(x) \leq g(x)$ for $\forall x \in [a, b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$
9. if $m \leq f(x) \leq M$ for $\forall x \in [a, b] \Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$

Note: $\int_a^b f(x)g(x)dx \neq \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$.

Fundamental Theorem of Calculus

FTC Part 1: If f is continious on $[a,b]$ then the function defined by $g(x) = \int_a^x f(t)dt, \quad a \leq x \leq b$; is continiuos on $[a,b]$, is differentiable on (a,b) and $g'(x) = f(x)$ (g is an antiderivative of f).

FTC Part 2: If f is continious on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

Notice that **FTC** suggests that differentiation and integration are inverse operations:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

$$\int F'(x)dx = F(x).$$

Techniques of Integration

Substitution Rule

For indefinite integrals:
If $u = g(x)$ is a differentiable function having range on interval I and f is continious on I then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

For definite integrals:
If g' is continious on $[a,b]$ and f is continious on the range of $u = g(x)$ then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Note: it is permissable to operate with dx and du after the integral signs as if they were differentials ($dg(x) = g'(x)dx$ or $du = \frac{du}{dx}dx$; $\frac{du}{dx} - derivative$).

Integration by Parts

For indefinite integrals:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or equivalently taking $u = f(x), v = g(x)$; $\int udv = uv - \int vdu$.

For definite integrals:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx.$$

Trigonometric Integrals

Strategy for evaluating $\int \sin^m(x)\cos^n(x)dx$:

a) if $n = 2k+1$ (the power of *cosine* is odd)

$$\int \sin^m(x) cos^n(x)dx = \int \sin^m(x)(1 - sin^2(x))^k \cos(x)dx$$

make substitution $u = \sin x$.

b) if $m = 2k+1$ (the power of *sine* is odd)

$$\int \sin^m(x) cos^n(x)dx = \int (1 - cos^2(x))^k \cos^n(x)\sin(x)dx$$

make substitution $u = \cos x$.

c) if both n and m are even, then use power reduction formulas (half-angle identities):

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x)) \quad \cos^2 x = \frac{1}{2}(1 + \cos(2x)).$$

Strategy for evaluating $\int \tan^m(x)\sec^n(x)dx$:

a) if $n = 2k$ (the power of *secant* is even)

$$\int \tan^m(x)\sec^n(x)dx = \int \tan^m(x)(1 + \tan^2(x))^{k-1}\sec^2(x)dx$$

make substitution $u = \sec x$.

b) if $m = 2k+1$ (the power of *tangent* is odd)

$$\int \tan^m(x)\sec^n(x)dx = \int (\sec^2(x) - 1)^k \sec^{n-1}(x)\tan(x)\sec(x)dx$$

make substitution $u = \tan x$.

c) if n is odd and m is even - no standart methods.

Strategy for $a)$ $\int \sin(mx)\cos(nx)dx$, $b)$ $\int \sin(mx)\sin(nx)dx$, $c)$ $\int \cos(mx)\cos(nx)dx$:

a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)$.

b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

Trigonometric Substitution

Use the following trigonometric substitutions for the corresponding expressions:

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$;	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$(a^2 - a^2\sin^2(x) = a^2\cos^2(x))$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$;	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$(a^2 + a^2\tan^2(x) = a^2\sec^2(x))$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$;	$0 \le \theta < \frac{\pi}{2}$	$(a^2\sec^2 - a^2 = a^2\tan^2(x))$

Improper Integrals

Improper Integrals of Type 1:

Definition:

a) If $\int_a^t f(x)dx$ exists for every number $t \ge a$, then define $\displaystyle\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$, provided the limit exists (as a finite number).

b) If $\int_t^b f(x)dx$ exists for every number $t \le b$, then define $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$, provided the limit exists (as a finite number).

c) if $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are both defined (corresponding limits exist as finite numbers), then we define $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$ (any number a can be used).

In these definitions the improper integrals are called *convergent* if the corresponding limits exist(as finite numbers), and *divergent* otherwise.
Note: for $c)$ both integrals in $a)$ and $b)$ must be convergent.

Improper Integrals of Type 2:

Definition:

a) If f is continious on $[a,b)$ and has a (infinite) discontinuity at b then define $\int_a^b f(x)dx = \lim_{t\to b-} \int_a^t f(x)dx$, provided the limit exists (as a finite number).

b) If f is continious on $(a,b]$ and has a (infinite) discontinuity at a then define $\int_a^b f(x)dx = \lim_{t\to a+} \int_t^b f(x)dx$, provided the limit exists (as a finite number).

c) If f has a (infinite) discontinuity at c where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are defined (corresponding limits exist as finite numbers), then we define $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

In these definitions the improper integrals are called *convergent* if the corresponding limits exist (as finite numbers), and *divergent* otherwise.
Note: for c) both integrals in a) and b) must be convergent.

Curves Defined by Parametric Equations

Tangents

If the curve $y = F(x)$, $a \le x \le b$ is also given by parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, $\alpha \le t \le \beta$; then the following fomula for the tangent to the curve in terms of the parametric equations holds:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \tfrac{dx}{dt} \neq 0 \qquad \text{or} \quad F'(x) = \frac{g'(t)}{f'(t)} \quad \text{if } f'(t) \neq 0.$$

If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0 \longrightarrow$ horizontal tangent.
If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0 \longrightarrow$ vertical tangent.
(The case $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ must be handled with more care considering $\lim(\frac{dy}{dt})/(\frac{dx}{dt})$).
Similarly for the second derivative one has:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}.$$

Note:

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}.$$

Areas

If the curve $y = F(x),\ a \leq x \leq b$ is also given by parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \alpha \leq t \leq \beta;$ and $a = f(\alpha), b = f(\beta)$, then the area under the curve over the interval $[a, b]$ is:

$$A = \int_a^b F(x)dx = \int_\alpha^\beta g(t)f'(t)dt.$$

Arc Length

Similar to areas under curves, one can procede with the definition of *arc length* of a curve by dividing the interval on which the curve is defined into subintervals, and taking the limit of the lengths of approximating polygons, having vertices at the points on the curve corresponding to the endpoints of subintervals.

If the curve C is given by its parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \alpha \leq t \leq \beta;$ both f' and g' are continious on $[\alpha, \beta]$ (and C is traversed exactly once as t increases from α to β then the length of C in terms of parametric equations is given by:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Polar Coordinates

Change of coordinates from polar coordinates to cartesian coordinates $[(r, \theta) \rightarrow (x, y)]$ is given by the following formulas:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

Change of coordinates from cartesian coordinates to polar coordinates $[(x, y) \to (r, \theta)]$ is given by the following formulas:

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} .$$

Tangents to polar curves

If the curve $y = F(x)$ is also given in terms of polar coordinates $r = f(\theta)$, then the tangent to the curve (in terms of polar coordinates) is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \quad \text{or} \quad F'(x) = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Areas of polar regions

The same process of approximation is employed in the definition of the area of polar regions, with the only difference of taking circular segments instead of approximating rectangles.

The area of a polar region $\mathcal{R}$, bounded by the curve $r = f(\theta)$ and rays $\theta = a$ and $\theta = b$ is given by:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta \quad \text{or} \quad A = \int_a^b \frac{1}{2}r^2 d\theta.$$

Arc Length

By treating θ as a parameter in the formula for the *arc length*, one easily finds that the length of a polar curve $r = f(\theta)$ with $a \le \theta \le b$ is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{or} \quad L = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta.$$