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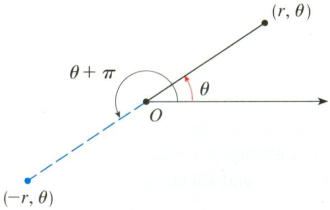

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Plotting points

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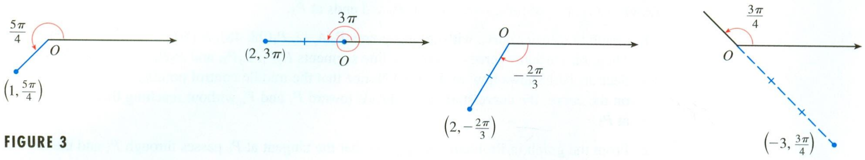
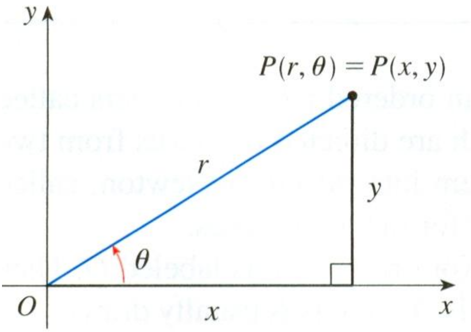


FIGURE 3

Coordinate conversion - Polar/Cartesian



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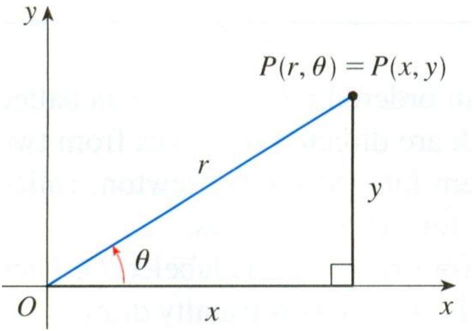

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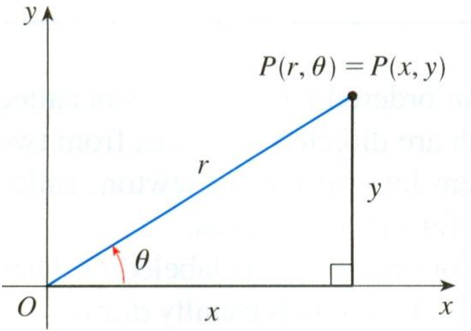

$$\mathbf{x} = \mathbf{r}\cos\theta \quad \mathbf{y} = \mathbf{r}\sin\theta$$

$$\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 \quad \tan\theta = \frac{\mathbf{y}}{\mathbf{x}}$$

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Therefore, the point is $(1, \sqrt{3})$ in **Cartesian** coordinates.

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Since the point $(1, -1)$ lies in the fourth quadrant, we choose $\theta = -\frac{\pi}{4}$ or $\theta = 7\,\frac{\pi}{4}$. Thus, one possible answer is $\left(\sqrt{2}, -\frac{\pi}{4}\right)$; another is $\left(\sqrt{2}, 7\,\frac{\pi}{4}\right)$.

Definition The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

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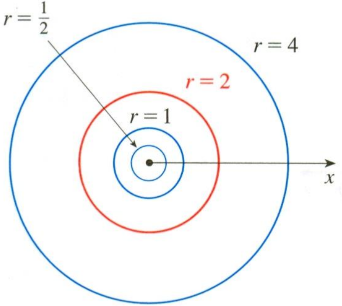

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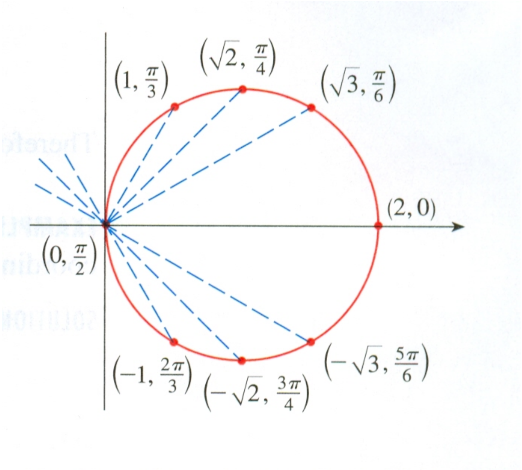

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This is the equation of a circle of radius 1 centered at $(1, 0)$.

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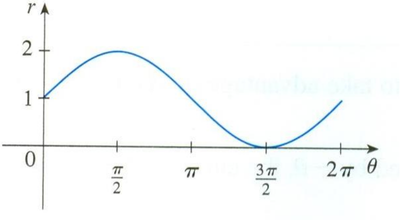

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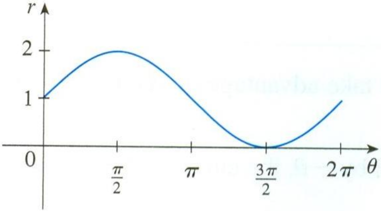

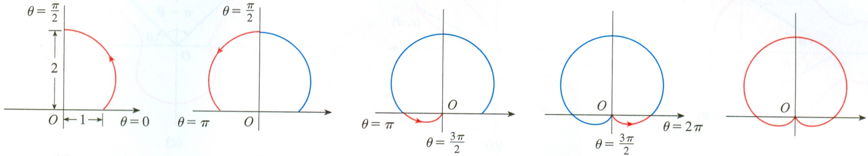

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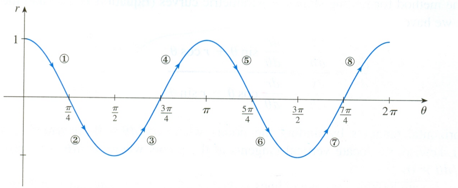

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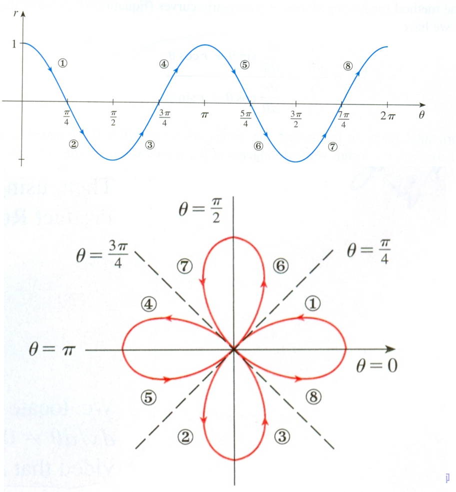

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$$= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

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The **slope of the tangent** at the point where $\theta = \frac{\pi}{3}$ is

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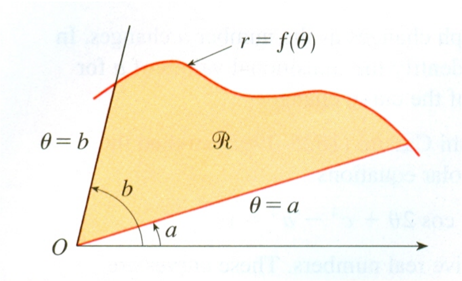
Also see the two figures below.

Area under a polar graph $\mathbf{r} = \mathbf{f}(\theta)$



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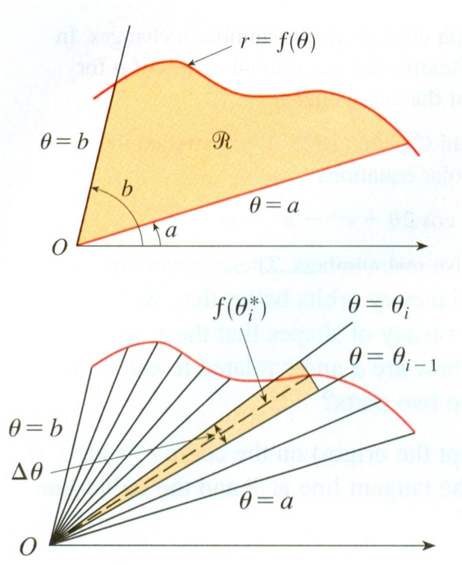

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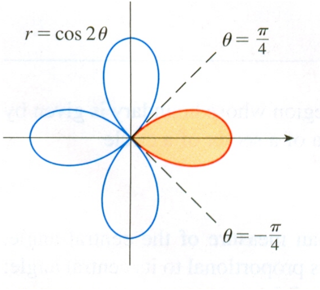

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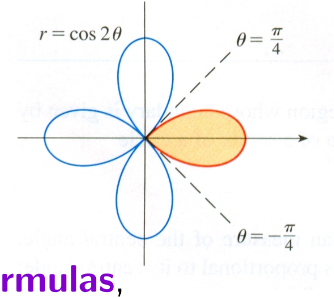

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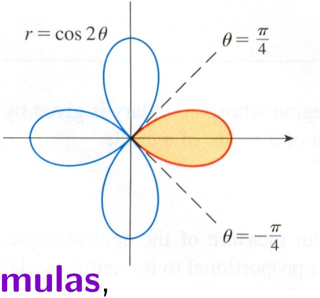

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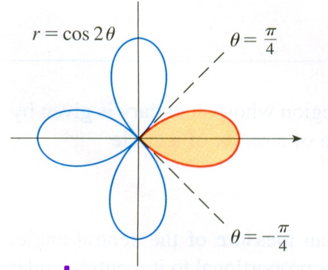

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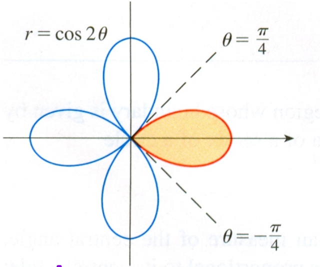

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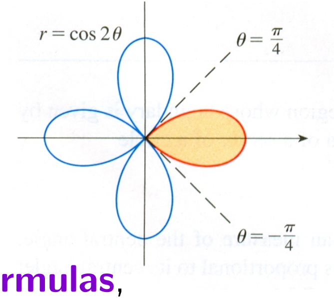

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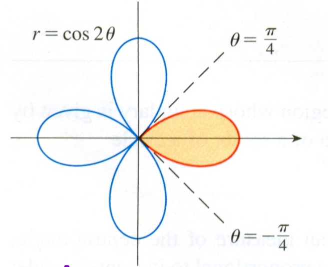

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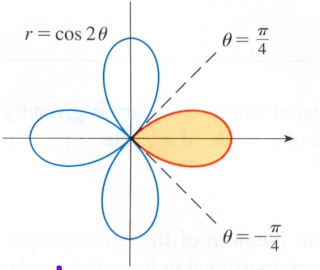

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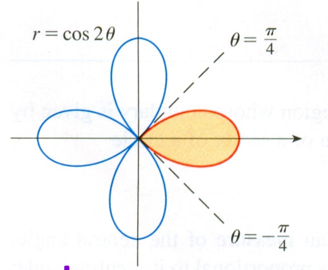

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Problem 30 As the parameter t increases forever, starting at $t = 0$, the curve with parametric equations

$$\begin{cases} x = e^{-t}\cos t, \\ y = e^{-t}\sin t \end{cases}$$

spirals inward toward the origin, getting ever closer to the origin (but never actually reaching) as $t \to \infty$.

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Thus the **length L** of a polar curve $\mathbf{r} = \mathbf{f}(\theta)$, $a \leq \theta \leq b$, is:

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Thus the **length L** of a polar curve $\mathbf{r} = \mathbf{f}(\theta)$, $a \le \theta \le b$, is:

$$\mathbf{L} = \int_{\mathbf{a}}^{\mathbf{b}} \sqrt{\mathbf{r}^2 + \left(\frac{\mathbf{dr}}{\mathbf{d}\theta}\right)^2}\ \mathbf{d}\theta.$$