MATH 552 **Numerical Linear Algebra** **Spring 2006**

Homework Set 5 Due Tuesday, 2 May 2006

1. Problem **10.2** from *Trefethen & Bau*.
2. Problem **11.3** from *Trefethen & Bau*.
3. Problem **12.2** from *Trefethen & Bau*.
4. Problem **13.3** from *Trefethen & Bau*.
5. Prove the following:

   (a) $\kappa(A) \geq 1$ for any induced matrix norm.

   (b) If U is unitary, then $\kappa_2(U) = 1$, $\kappa_2(UA) = \kappa_2(AU) = \kappa_2(A)$.

   (c) $\kappa_2(A) = \sigma_{\max}/\sigma_{\min}$.

   (d) If A is hermitian, then $\kappa_2(A) = |\lambda|_{\max}/|\lambda|_{\min}$.

   (e) If $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, then $\dfrac{\|\delta x\|/\|x + \delta x\|}{\|\delta A\|/\|A\|} \leq \kappa(A)$.

6. Consider $Ax = b$. Let x be the exact solution and let $\tilde{x}$ be an approximate solution. The **error** is $e = x - \tilde{x}$ and the **residual** is $r = b - A\tilde{x}$.

   (a) Show that $Ae = r$ and $\dfrac{\|e\|}{\|x\|} \leq \kappa(A)\dfrac{\|r\|}{\|b\|}$.

   (b) It follows that if A is invertible, then $e = 0$ if and only if $r = 0$, but if A is ill-conditioned, then the relative error $\dfrac{\|e\|}{\|x\|}$ may be large even if the relative residual $\dfrac{\|r\|}{\|b\|}$ is small. This occurs in the following example (due to W. Kahan).

   $$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix},\ b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix},\ x = \begin{pmatrix} 2 \\ -2 \end{pmatrix},\ \tilde{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \tilde{x}_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}$$

   Show that $Ax = b$ (using exact arithmetic). Consider $\tilde{x}_1$ and $\tilde{x}_2$ as approximate solutions and or each one compute the corresponding $\dfrac{\|e\|_\infty}{\|x\|_\infty}$, $\dfrac{\|r\|_\infty}{\|b\|_\infty}$. Find $\kappa_\infty(A)$.