HW #3 Solutions: M552 Spring 2006

1. (**4.1**-*Trefethen & Bau*: parts (a), (c) and (e)) Determine the SVDs of the following matrices (by hand calculation):

$$\text{(a)} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(e)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

ANS: In each case we seek $A = U\Sigma V^*$. The general algorithm is to first find $A^*A = (U\Sigma V^*)^* U\Sigma V^* = V\Sigma^*\Sigma V^* = VDV^*$, the spectral decomposition of the hermitian matrix A^*A. Then take $\Sigma = \sqrt{D}$, noting that Σ may have to be extended to the size of A, i.e. zero rows appended, and solve for U from $AV = U\Sigma$. However, if $A = A^*$, then we can use A's spectral decompostion $A = PDP^* = P|D|\text{sign}(D)P^* = U\Sigma V^*$, where $U = P$, $\Sigma = |D|$ and $V^* = \text{sign}(D)P^*$.

(a): $A^* = A$, and in fact A is already in diagonal form. So $D = A$ and $P = I$. Then

$$A = PDP^* = IDI = I * |D| * sign(D)I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = U\Sigma V^*.$$

(c): Following the general algorithm, $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = I * \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} * I^*$, the spectral decompostion of A^*A. Note that the diagonal entries of D are not decreasing. We can achieve this by interchanging the columns of I and the diagonal entries of D, which simply amounts to reordering the e-pairs of A^*A, giving $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then extending $\sqrt{\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Now choose U unitary so that $AV = U\Sigma$, or $\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [u_1\ u_2\ u_3] \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. We see that this gives $U = I$. Thus,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^*.$$

(e): $A^* = A$, and it is easy to see that the e-values of A are $\lambda_1 = 2$ and $\lambda_2 = 0$, and corresponding mutually orthogonal e-vectors $p_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $p_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (do the computation if you don't see this). Then

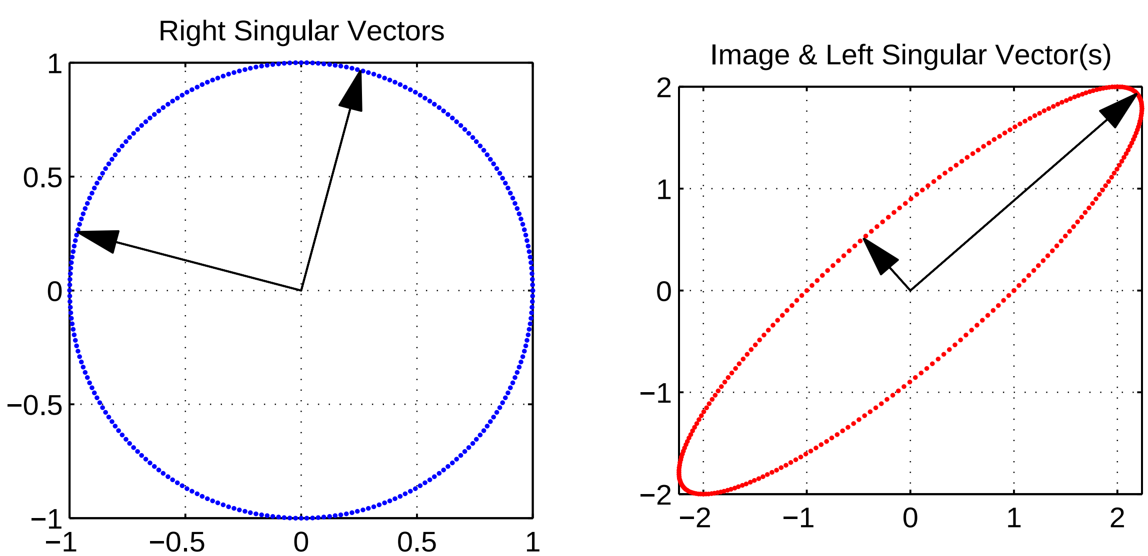
$$A = PDP^* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^*.$$

2. (**4.3**-*Trefethen & Bau*) Write a MATLAB program which, given a real 2×2 matrix A, plots the right singular vectors v_1 and v_2 in the unit circle and also the left singular vectors u_1 and u_2 in the appropriate ellipse, as in Figure 4.1. Apply your program to the matrix (3.7), $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, and also the 2×2 matrices of exercise 4.1. You may use MATLAB's `svd` command to generate the singular vectors and values.

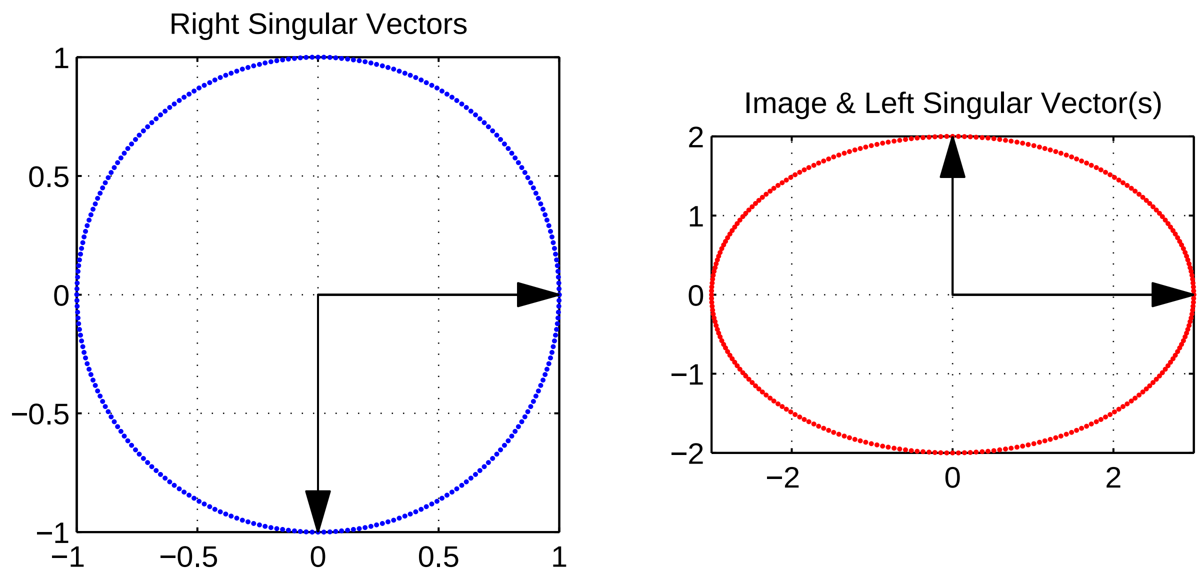
<u>**ANS**</u>: Here is the code for a given 2×2 matrix A:

```
[U,S,V]=svd(A); S=diag(S);  % find SVD

th=0:2*pi/256:2*pi;
dom=[cos(th); sin(th)]; ran=A*dom; % define points on unit circle and image

subplot(1,2,1)
plot(dom(1,:),dom(2,:),'.'),axis('image'),grid,hold
arrow([0,0],[V(1,1),V(2,1)])
arrow([0,0],[V(1,2),V(2,2)])
title('Right Singular Vectors')

subplot(1,2,2)
plot(ran(1,:),ran(2,:),'r.'),axis('image'),grid,hold
arrow([0,0],S(1)*[U(1,1),U(2,1)])
if (abs(S(2)) > 10e-10)           % check that singular value is nonzero
arrow([0,0],S(2)*[U(1,2),U(2,2)])
end
title('Image & Left Singular Vector(s)')

% arrow.m is a user-contributed M-file available at www.mathworks.com
```

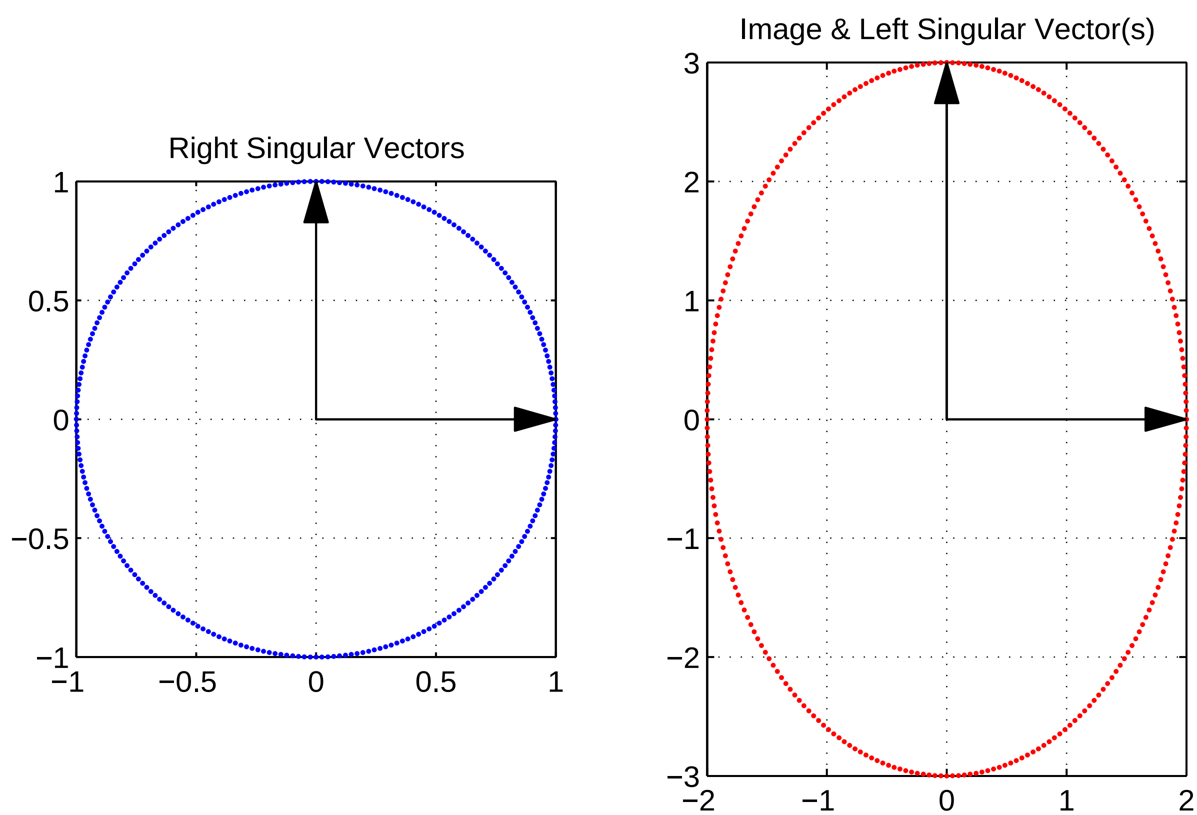
Results for $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$:



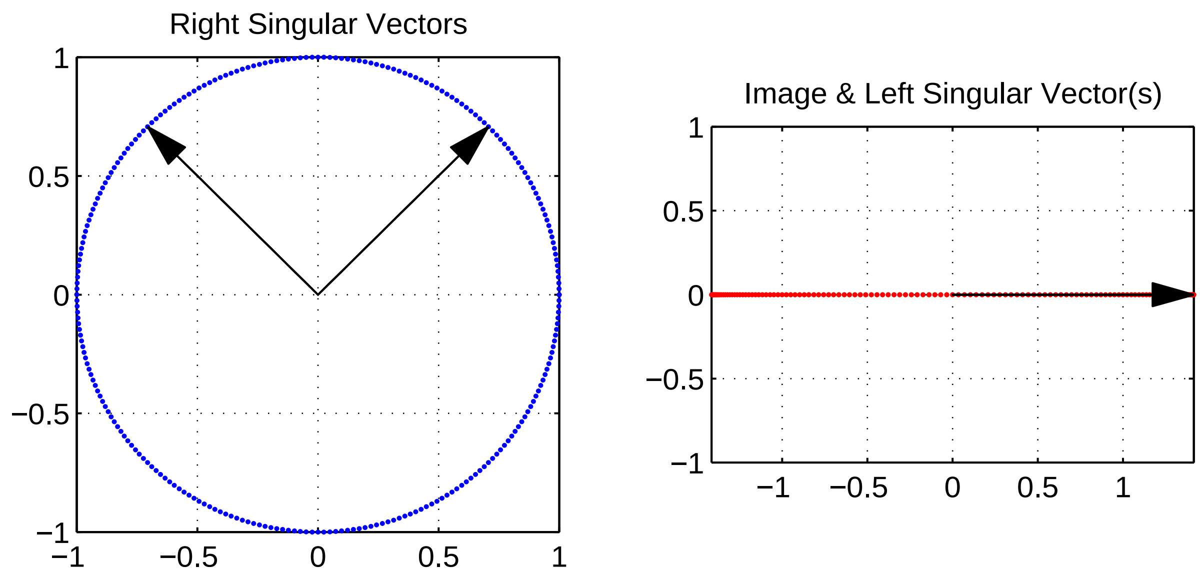
Results for $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$:



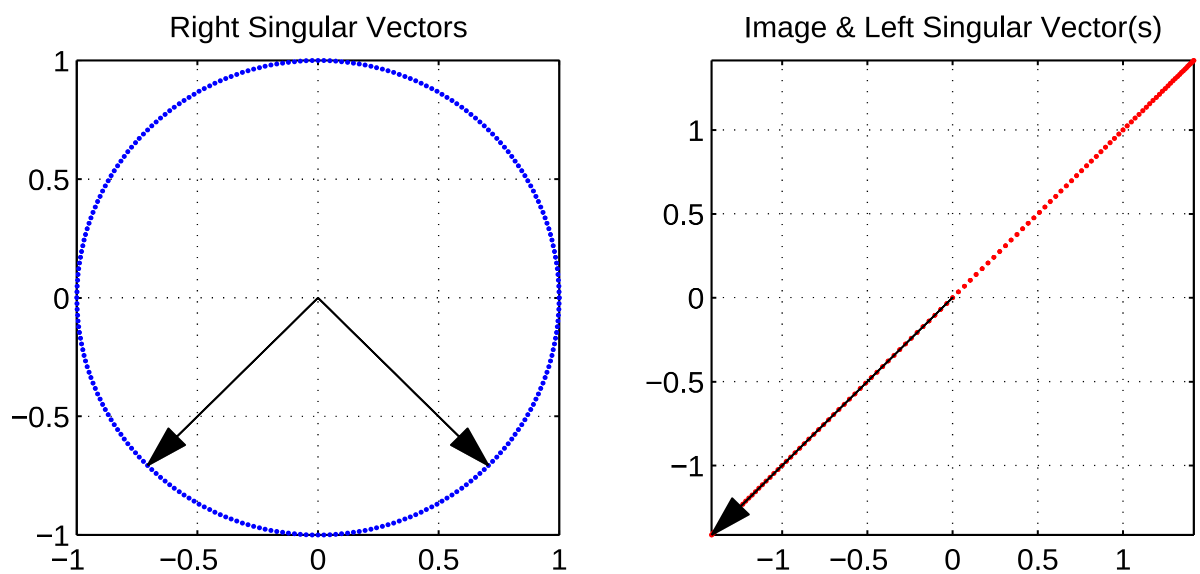
Results for $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$:



Results for $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$:



Results for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$:



3. (**4.5**-*Trefethen & Bau*) Theorem 4.1 asserts that every $A \in \mathbb{C}^{m\times n}$ has an SVD $A = U\Sigma V^*$. Show that is A is real, then it has a real SVD ($U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$).

   ANS: If $A \in \mathbb{R}^{m\times n}$ then $A^*A = A^TA \in \mathbb{R}^{n\times n}$ is symmetric, hence $A^TA = VDV^T$ where $D \in \mathbb{R}^{n\times n}$ is a real diagonal matrix, with non-negative entries (we proved this in class), and $V \in \mathbb{R}^{n\times n}$ is a real orthogonal matrix. Hence $\Sigma = \sqrt{D} \in \mathbb{R}^{n\times n}$ is defined. If $m > n$ we can append zero rows to Σ to ensure $\Sigma \in \mathbb{R}^{m\times n}$. What if $m < n$? Think about it!

   We can solve for U using $AV = U\Sigma$, showing that U can be chosen to ensure $U \in \mathbb{R}^{m\times m}$ since A, Σ and V are all real matrices.

4. (**5.1**-*Trefethen & Bau*) In Example 3.1 we considered the matrix (3.7), $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, and asserted, among other things, that its 2-norm is approximately 2.9208. Using the SVD, work out (on paper) the exact values of $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ for this matrix.

<u>**ANS**</u>: To find the singular values, form $A^*A = (U\Sigma V^*)^* U\Sigma V^* = V\Sigma^*\Sigma V^* = VDV^*$, the spectral decomposition of the hermitian matrix A^*A. The e-values are the diagonal entries of D, and $\Sigma = \sqrt{D}$.

$$A^*A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \quad \Rightarrow \quad |A^*A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{bmatrix} \right| = \lambda^2 - 9\lambda + 4,$$

whose roots are $\lambda_{1,2} = \dfrac{9 \pm \sqrt{(-9)^2 - 4*1*4}}{2} = \dfrac{9 \pm \sqrt{65}}{2}$. So we have

$$\sigma_{\min}(A) = \sqrt{\frac{9-\sqrt{65}}{2}} \approx 0.6847, \quad \sigma_{\max}(A) = \sqrt{\frac{9+\sqrt{65}}{2}} \approx 2.9208,$$

giving $\|A\|_2 = \sigma_{\max}(A) \approx 2.9208$.

5. (**5.4**-*Trefethen & Bau*) Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition (5.1), $X\Lambda X^{-1}$, of the $2m \times 2m$ hermitian matrix $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$.

<u>**ANS**</u>: Working on it!

6. Find the eigenvalues and a corresponding eigenfunction for the operator $-d^2/dx^2$ applied to functions with homogeneous boundary conditions on $[0,1]$, i.e., solutions of

$$-y'' = \lambda y, \quad y(0) = y(1) = 0.$$

Note that there are a infinite number of them, say $(\lambda_k, y_k(x))$ for $k = 1, 2, \ldots$, with $0 < \lambda_1 < \lambda_2 < \ldots$.

ANS: We have to solve a second order constant coefficient ODE, whose solution must also satisfy the given boundary conditions (BCs). We have

$$y'' + \lambda y = 0 \quad \Rightarrow \quad r^2 + \lambda = 0 \quad \Rightarrow \quad r_{1,2} = \pm\sqrt{-\lambda}.$$

The specific form of the solution depends on the sign of λ as follows:

$\underline{\lambda < \mathbf{0}}$: So $-\lambda > 0$. The general solution is given by $y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. Applying the BCs,

$$0 = y(0) = C_1 e^{\sqrt{-\lambda}0} + C_2 e^{-\sqrt{-\lambda}0} = C_1 + C_2,$$

$$0 = y(1) = C_1 e^{\sqrt{-\lambda}1} + C_2 e^{-\sqrt{-\lambda}1} = C_1 e^{\sqrt{-\lambda}} + C_2 e^{-\sqrt{-\lambda}},$$

or in matrix form

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}} & e^{-\sqrt{-\lambda}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \text{since} \quad \left| \begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}} & e^{-\sqrt{-\lambda}} \end{bmatrix} \right| = e^{-\sqrt{-\lambda}} - e^{\sqrt{-\lambda}} \neq 0,$$

so the system has only the trivial solution, thus $y(x) \equiv 0$ which can't be an eigenfunction.

$\underline{\lambda = \mathbf{0}}$: The general solution is given by $y(x) = C_1 + C_2 x$. Applying the BCs,

$$0 = y(0) = C_1 + C_2 * 0 = C_1 \quad \Rightarrow \quad C_1 = 0,$$

$$0 = y(1) = C_2 * 1 = C_2 \quad \Rightarrow \quad C_2 = 0.$$

So again, we arrive at the trivial solution $y(x) \equiv 0$ which can't be an eigenfunction.

$\underline{\lambda > \mathbf{0}}$: So $-\lambda < 0$ which gives rise to complex roots $r_{1,2}$. In this case the general real solution is given by $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. Applying the BCs,

$$0 = y(0) = C_1 \cos\sqrt{\lambda}0 + C_2 \sin\sqrt{\lambda}0 = C_1 * 1 + C_2 * 0 = C_1 \quad \Rightarrow \quad C_1 = 0,$$

$$0 = y(1) = C_2 \sin\sqrt{\lambda}1 = C_2 \sin\sqrt{\lambda}.$$

Choosing $C_2 = 0$ gives $y(x) \equiv 0$, the trivial solution. For a nontrivial solution we must have $\sin\sqrt{\lambda} = 0$, which is satisfied by an infinite number of $\lambda > 0$, given by (note $\sqrt{\lambda} > 0$)

$$\sqrt{\lambda_k} = k\pi \quad \text{for} \quad k = 1, 2, \ldots, \quad \Rightarrow \quad \lambda_k = (k\pi)^2 \quad \text{for} \quad k = 1, 2, \ldots.$$

These $\{\lambda_k\}_{k=1}^{\infty}$ are thus the eigenvalues of the continuous second derivative operator for functions defined on $[0,1]$ with homogeneous BCs. For each k, taking $C_2 = 1$, a corresponding eigenfunction is given by $y_k(x) = \sin\sqrt{\lambda_k}x = \sin k\pi x$.

Summarizing, the e-pairs are given by

$$(\lambda_k, y_k(x)) = ((k\pi)^2, \sin k\pi x) \text{ for } k = 1, 2, \ldots, \text{ and note } 0 < \lambda_1 < \lambda_2 < \ldots \text{ with } \lim_{k\to\infty} \lambda_k = \infty.$$

7. Consider the discrete analog of the eigenproblem in the previous problem, with $-d^2/dx^2 \approx -D_+D_- = -D^2$. To this end, choose $N > 0$, let $h = 1/N$ and $x_i = i * h$ for $i = 0, 1, \ldots, N$. Note we now have $N+1$ grid points with $x_0 = 0$ and $x_N = 1$. So we seek e-pairs which satisfy

$$(-D^2u)_i = \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \lambda u_i \quad \text{for } i = 1, 2, \ldots, N-1, \quad u_0 = u_N = 0,$$

or in matrix form,

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}$$

(a) Show that (λ_k, u_k) is an e-pair for $k = 1, 2, \ldots, N-1$ where $\lambda_k = 2(1 - \cos k\pi h)/h^2$ and u_k is the vector with components $(u_k)_i = \sin ik\pi h$. Hint: Use trig identities, and note that $\sin 0k\pi h = \sin Nk\pi h = 0$.

(b) For each $N = 8$ and $N = 16$ plot graphs (using *subplot*) of the continuous eigenfunctions found in the previous problem, and the discrete eigenvectors from (a) for $k = 1, 2, N/2$ and $N-1$. Thus each plot should have 4 images. Use a linetype for the continuous eigenfunctions and symbols for the discrete eigenvectors. Label and title your graphs.

(c) In a single plot display the first 32 eigenvalues vs. there number, i.e., $k = 1, 2, \ldots, 32$ of the continuous problem along with the $N-1$ eigenvalues for each $N = 8, 16$ and 32. For the continuous case plot the eigenvalues using a linetype with a symbol, and in the discrete case just symbols. Label, title, and place a legend on your plot.

Discuss the results.

<u>ANS</u>:

(a) Noting that $\sin 0k\pi h = \sin Nk\pi h = 0$. Fix a k, $k = 1, \ldots, N-1$. For $i = 1, \ldots, N-1$,

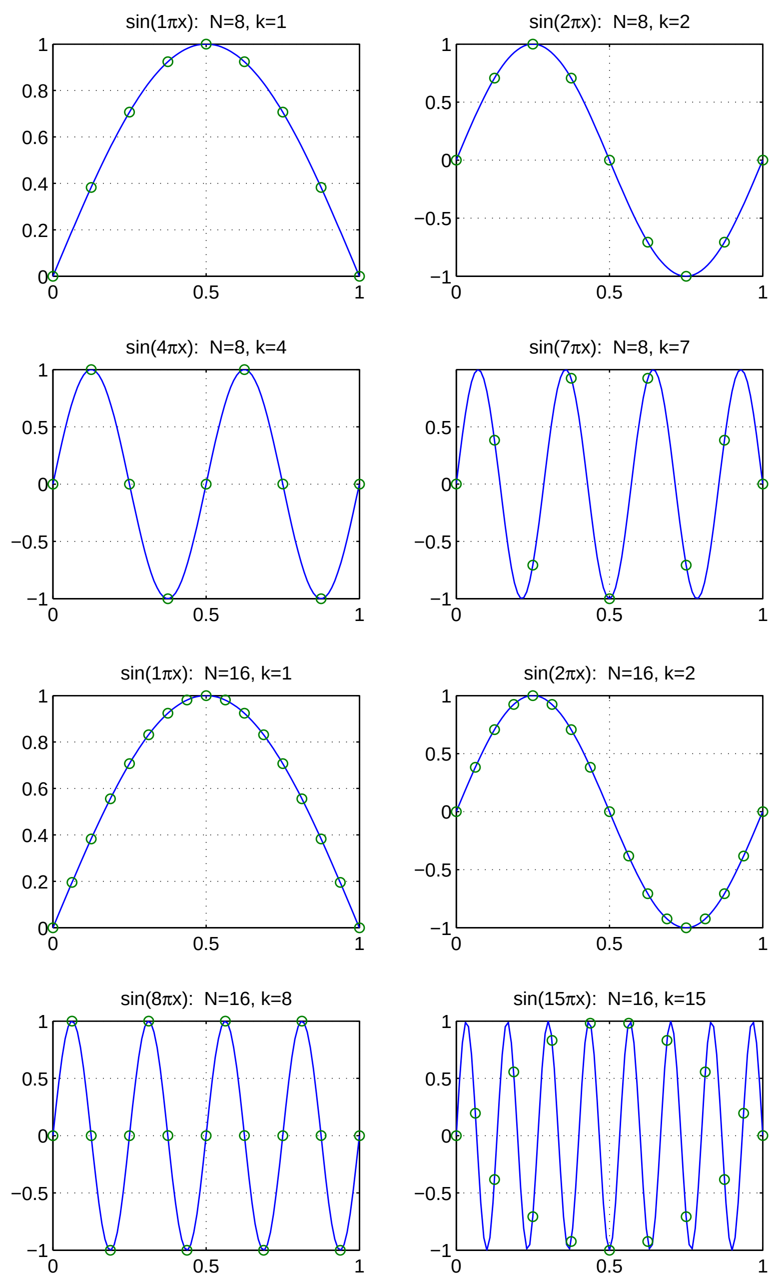
$$\begin{aligned}
\frac{-u_{k,i-1} + 2u_{k,i} - u_{k,i+1}}{h^2} &= \frac{-\sin(i-1)k\pi h + 2\sin ik\pi h - \sin(i+1)k\pi h}{h^2} \\
&= \frac{-(\sin ik\pi h \cos k\pi h - \cos ik\pi h \sin k\pi h) + 2\sin ik\pi h - (\sin ik\pi h \cos k\pi h + \cos ik\pi h \sin k\pi h)}{h^2} \\
&= \frac{-2\sin ik\pi h \cos k\pi h + 2\sin ik\pi h}{h^2} \\
&= \frac{2(1 - \cos k\pi h)}{h^2} \sin ik\pi h \\
&= \lambda_k \sin ik\pi h. \\
&= \lambda_k u_{k,i}.
\end{aligned}$$

So we have $Au_k = \lambda_k u_k$ for $k = 1, \ldots, N-1$, where A is the matrix representing $-D^2$. Thus, (λ_k, u_k) are e-pairs for $k = 1, \ldots, N-1$.

(b) Here is the code for $N = 8$:

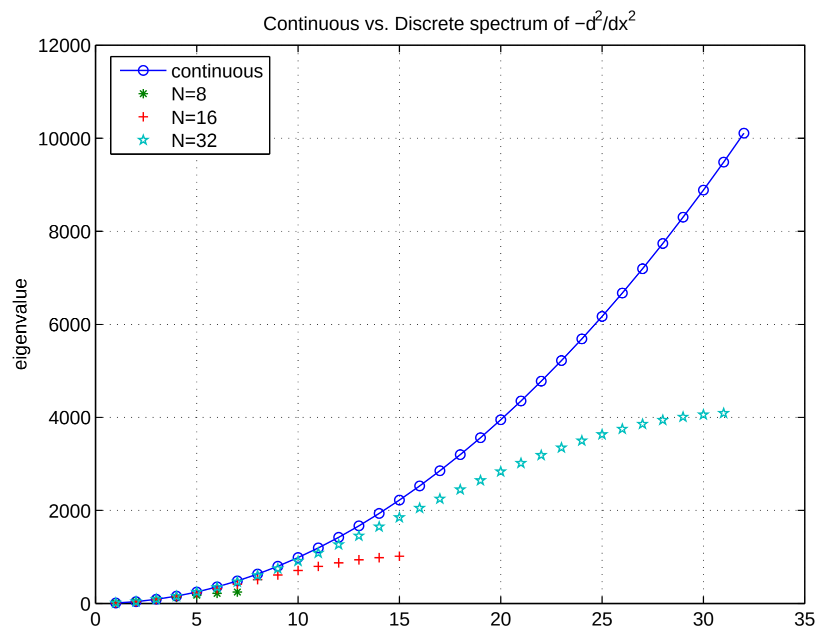
```
N=8;  h=1/N; x=0:h:1; xx=0:0.01:1; k=[1 2 N/2 N-1];
for i=1:4
    subplot(2,2,i)
    plot(xx,sin(k(i)*pi*xx),x,sin(k(i)*pi*x),'o')
    grid,axis('tight')
    title(['sin(',num2str(k(i)),'\pix):  N=',num2str(N),', k=',num2str(k(i))]);
end
```

And the graphs for $N = 8$ and $N = 16$:



(c) Here is the code to generate the plot, and the plot itself:

```
N=[8 16 32]; h=1./N; N=N-1;
plot(1:32,(pi*(1:32)).^2,'-o',...
     1:N(1),2*(1-cos((1:N(1))*pi*h(1)))/(h(1)^2),'*',...
     1:N(2),2*(1-cos((1:N(2))*pi*h(2)))/(h(2)^2),'+',...
     1:N(3),2*(1-cos((1:N(3))*pi*h(3)))/(h(3)^2),'p')
grid,title('Continuous vs. Discrete spectrum of -d^2/dx^2')
ylabel('eigenvalue'),legend('continuous','N=8','N=16','N=32',2)
```



Discussion: From part (b) we see that the eigenvectors of $-D^2$ are the corresponding eigenfunctions of the continuous problem evaluated at the grid points determined by the specific N. In this sense the discrete problem is clearly a very good approximation to the continuous one. On the other hand, note in the plot above that while for each N the first few discrete eigenvalues are good approximations, this is not the case for the eigenvalues corresponding to the higher modes. Thus one expects these modes in the continuous problems not to be modeled well by the discrete approximation.

Note: A Taylor series expansion gives

$$\begin{aligned}
(k\pi)^2 - \lambda_k &= (k\pi)^2 - \frac{2}{h^2}(1 - \cos k\pi h) \\
&= (k\pi)^2 - \frac{2}{h^2}(1 - (1 - \frac{(k\pi h)^2}{2} + \frac{(k\pi h)^4}{24} - \cdots)) \\
&= (k\pi)^2 - \frac{2}{h^2}\left(\frac{(k\pi h)^2}{2} - \frac{(k\pi h)^4}{24} + \cdots\right) = (k\pi)^2 - (k\pi)^2 + \frac{2}{h^2}\left(\frac{(k\pi h)^4}{24} - \cdots\right) \\
&= \left(\frac{(k\pi)^4}{12}h^2 - O(h^4)\right) = O(h^2)
\end{aligned}$$

So we do have the expected accuracy of approximation, but remember there is still a term in the error that depends on the eigenvalue being approximated, $(k\pi)^4/12$, which explains, even for larger N, why we do not see rapid convergence of the discrete eigenvalues to the continuous ones as $k \to N-1$. In fact, for a fixed N and k large, we see divergence in the upper spectrum.