MATH 552 **Scientific Computation II** **Spring 2020**

Homework Set 5 Due Friday, 17 April 2020

1. *Orthogonality of complex exponentials*: Consider the complex exponential functions
$$\phi_n(x) = e^{-i(\frac{n\pi x}{L})} \quad \text{for} \quad -\infty < n < \infty\,.$$
Show that
$$\langle \phi_n(x), \phi_m(x)\rangle = \int_{-L}^{L} \phi_n(x)\overline{\phi_m}(x)\ dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } n = m\,, \end{cases}$$
and thus the functions are mutually orthogonal.

2. *Fourier Series and Orthogonality of Sines and Cosines:* The Fourier series for $f \in L^2[-\pi, \pi]$, given by
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx\,,$$
is an expansion of the function $f(x)$ in the basis of trigonometric functions
$$\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \ldots\} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$
   (a) Recall the inner product
   $$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\ dx\,,$$
   find an expression for $\langle \phi_n(x), \phi_m(x)\rangle$ for $m, n \geq 0$. In particular, show that the basis functions are mutually orthogonal by showing that for $n \neq m$ the inner product is zero.

   (b) Derive concise expressions for the a_k and b_k in Fourier series for $f(x)$.

3. The method of undetermined coefficients was used to derived the 2^{nd} order centered finite difference approximation to both $u'(x_j)$ and $u''(x_j)$, given respectively by
$$(\widetilde{D}u)(x_j) = \frac{u_{j+1} - u_{j-1}}{2h}, \quad \text{and} \quad (D^2u)(x_j) = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}.$$
Here $u_j = u(x_j)$ and $u_{j\pm1} = u(x_j \pm h)$ where $x_{j\pm1} = x \pm h$.

   (a) Show that the long stencil centered finite difference approximation to $u'(x_j)$,
   $$(\widetilde{D}(1 - \frac{h^2}{6}D^2)u)(x_j) = \frac{u_{j-2} - 8u_{j-1} + 0u_j + 8u_{j+1} - u_{j+2}}{12h}\,,$$
   is an $O(h^4)$ approximation to $u'(x_j)$. Derive a concise formula for the $O(h^4)$ error term.

   (b) We showed earlier in the term that $\widetilde{D} = \frac{d}{dx} + O(h^2)$. A more careful analysis gives
   $$\widetilde{D} = \frac{d}{dx} + \frac{h^2}{6}\frac{d^3}{dx^3} + O(h^4)\,.$$
   Use this expression to both (1) derive the operator form of the long stencil approximation $\widetilde{D}(1 - \frac{h^2}{6}D^2)$, and (2) in doing so show that it is indeed an $O(h^4)$ approximation to $\frac{d}{dx}$.

(c) Use the long stencil formula to approximate $u'(1)$, where $u(x) = e^x$, for $h = 2^{-N}$ for $N = 2 : 10$. Form a table with columns giving h, the approximation, absolute error and absolute error divided by h^4. For each indicate to which values they are converging. Finally, verify that the last column appears to be converging to a value derived using the error term.

4. Consider the 2-point BVP with Neumann BCs,

$$\begin{cases} -u'' + u = f \\ u'(0) = u'(1) = 0\,. \end{cases}$$

Let $u(x) = x^2(x-1)^2 e^x$ be the solution, from which you can derive $f(x)$.

(a) Write a MATLAB script to solve the problem by the FFT method, using the *Discrete Cosine Transform* as implemented by *dct.m* applied to the **2nd** order centered FD scheme, assuming $\sigma > 0$ is a **constant**,

$$-D^2 v_i + \sigma v_i = f_i\,.$$

Assume a meshsize $h = 1/2^p$, where p is a positive integer. For $p = 1 : 4$, plot the exact solution ($u(x)$ vs. x) and the numerical solution (v_i vs. x_i), including the boundary points. The 4 plots should appear separately in one figure, with axes labeled and a title for each indicating p. Investigate **subplot** in MATLAB for how to have multiple plots in a single figure window. Include a copy of your code.

(b) For $p = 1 : 10$ present a table with the following data - column 1: h; column 2: $\|u_h - v_h\|_\infty$; column 3: $\|u_h - v_h\|_\infty / h^2$, where $h = 1/n$. Discuss the trends in each column. Include a copy of your code.