Math 552 **Scientific Computing II** **Spring 2020**

SOLUTIONS: Homework Set 4

1. Consider the 2-point BVP

$$\begin{cases} -u'' + (4x^2 + 2)u = 2x(1 + 2x^2) \\ u(0) = 1,\ u(1) = 1 + e \end{cases}$$

   (a) Show $u(x) = x + e^{x^2}$ is the exact solution.

   (b) Write a MATLAB function M-file to solve the problem using the second order centered FD scheme we discussed in class, $-D^2 v_i + \sigma_i v_i = f_i$. Your code should use your M-files **trilu** and **trilu_solve**. Include a copy of your code.

   (c) For mesh sizes $h = (1/2)^p$, $p = 1 : 4$, plot the exact solution ($u(x)$ vs. x) and the numerical solution (v_i vs. x_i), including the boundary points. The 4 plots should appear separately in one figure, with axes labeled and a title for each indicating p. Investigate **subplot** in MATLAB for how to have multiple plots in a single figure window.

   (d) For mesh sizes $h = (1/2)^p$, $p = 1 : 10$ present a table with the following data - column 1: h; column 2: $\|u_h - v_h\|_\infty$; column 3: $\|u_h - v_h\|_\infty / h^2$, where $h = 1/n$. Discuss the trends in each column.

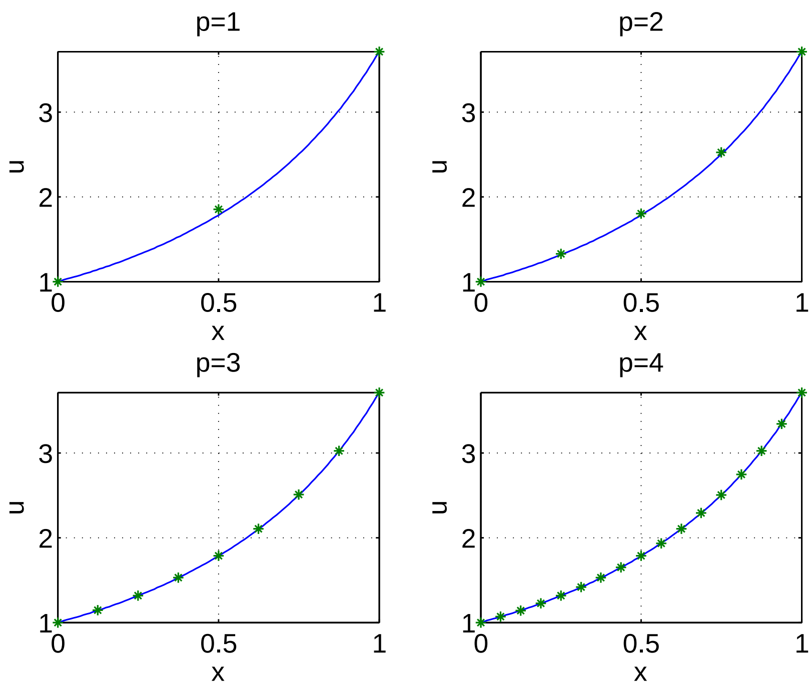
<u>**ANS**</u>: **(a)** This follows by simply substituting $u(x)$ into the ODE and also checking the boundary conditions.

(b) Here is a copy of the code:

```
function [xv,uv,stime] = fdbvp_2nd(n,a,b,ua,ub,sigma,f)
%
h = (b-a)/n;
xv = a:h:b; xv=xv';
fv = zeros(n+1,1);
sigmav = zeros(n+1,1);
for i = 1:n+1
    x = xv(i);
    sigmav(i) = eval(sigma);
    fv(i) = eval(f);
end
%
av = (2*ones(n-1,1)+h*h*sigmav(2:n))/(h*h);
bv = -ones(n-1,1)/(h*h); bv(1)=0;
cv = -ones(n-1,1)/(h*h); cv(n-1)=0;
fv = fv(2:n);
fv(1) = fv(1)+ua/(h*h); fv(n-1) = fv(n-1)+ub/(h*h);
%
tic;
[alpha,beta]=trilu(av,bv,cv);
```

```
uv=trilu_solve(alpha,beta,cv,fv);
stime=toc;
uv=[ua;uv;ub];
```

(c) Here is the graph:



(d)

```
        h           inf_err         err/h^2
  ----------------------------------------


   5.0000e-01   6.8077e-02   2.7231e-01
   2.5000e-01   2.0012e-02   3.2019e-01
   1.2500e-01   5.4792e-03   3.5067e-01
   6.2500e-02   1.3851e-03   3.5458e-01
   3.1250e-02   3.4860e-04   3.5697e-01
   1.5625e-02   8.7213e-05   3.5722e-01
   7.8125e-03   2.1807e-05   3.5729e-01
   3.9062e-03   5.4521e-06   3.5731e-01
   1.9531e-03   1.3630e-06   3.5731e-01
   9.7656e-04   3.4076e-07   3.5731e-01
```

We can see from the err/h^2 column that the expected $O(h^2)$ error is observed.

2. Find the eigenpairs $(\lambda, u(x))$ of the Laplacian with homogeneous Dirichlet BCs on the interval $[0, 1]$, i.e.

$$\begin{cases} u'' = \lambda u \\ u(0) = u(1) = 0 \end{cases}$$

ANS: (Hint) You must go through the 3 possibilities of $\lambda < 0, \lambda = 0$ and $\lambda > 0$, and find the general solution of the resulting second order constant coefficient ODE. Then apply the BCs to determine if there is a non-zero eigenfunction. The only case that will result in such is $\lambda < 0$.

3. Consider the linear operator $L = d/dx$ on the vector space $\mathcal{P}_2$, the set of all polynomials of degree less than or equal to 2. For each of the bases S below find the matrix A representation of L, i.e. $A = [L]_S$. Then for each find the eigenvalues of A.

   (a) $S_1 = \{1, x, x^2\}$

   (b) $S_2 = \{2, x+1, x^2-1\}$

   What is the relationship between the eigenvalues of A_{S_1} and A_{S_2}?

   ANS: Using the notation $S = \{u_1, u_2, u_3\}$. Note that $col_j(A) = [L(u_j)]_S$, the coordinates of $L(u_j)$ with respect to the S basis.

   For S_1 we have

$$L(u_1) = L(1) = 0 = 0 * u_1 + 0 * u_2 + 0 * u_3, \qquad L(u_2) = L(x) = 1 = 1 * u_1 + 0 * u_2 + 0 * u_3$$

$$L(u_3) = L(x^2) = 2x = 0 * u_1 + 2 * u_2 + 0 * u_3 \quad \text{so} \quad A_{S_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

   For S_2 we have

$$L(u_1) = L(2) = 0 = 0*u_1+0*u_2+0*u_3, \quad L(u_2) = L(x+1) = 1 = (1/2)*u_1+0*u_2+0*u_3$$

$$L(u_3) = L(x^2-1) = 2x = -1 * u_1 + 2 * u_2 + 0 * u_3 \quad \text{so} \quad A_{S_2} = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

   In both cases the eigenvalues of A_S – they are just the diagonal entries – are all 0, which are those of the operator L. They have to be as the eigenvalues are intrinsic to the operator.

4. Given a real vector $v = (v_1, \ldots, v_{N-1})^T$ the *Discrete Sine Transform* of v is given by $\hat{v} = P^{-1}v$, where P is an $(N-1) \times (N-1)$ matrix with $P_{i,j} = (2/\sqrt{2N}) \sin(ij\pi/N)$ for $i, j = 1, 2, \ldots, N-1$.

   Show that (a) $P = P^T$ and (b) $P^{-1} = P$. (Note - the sums are most easily computed by writing the sines in terms of complex exponentials.)

   ANS: **(a)** First, $P_{i,j} = (2/\sqrt{2N}) \sin(ij\pi/N) = (2/\sqrt{2N}) \sin(ji\pi/N) = P_{j,i} \quad \Rightarrow \quad P = P^T$.

   To show $P^{-1} = P$ we compute P^2. Suppose $l \neq j$, then

$$
\begin{aligned}
(P^2)_{l,j} &= \sum_{k=1}^{N-1} \frac{2}{\sqrt{2N}} \sin(lk\pi/N) \frac{2}{\sqrt{2N}} \sin(kj\pi/N) \\
&= \frac{1}{N} \sum_{k=1}^{N-1} 2 \sin(lk\pi/N) \sin(kj\pi/N) \\
&= \frac{1}{N} \sum_{k=1}^{N-1} [\cos(k(l-j)\pi/N) - \cos(k(l+j)\pi/N)] \\
&= \frac{1}{2N} \left[\sum_{k=1}^{N-1} 2\cos(k(l-j)\pi/N) - \sum_{k=1}^{N-1} 2\cos(k(l+j)\pi/N) \right] \\
&= \frac{1}{2N} \left[\sum_{k=1}^{N-1} (e^{i(k(l-j)\pi/N)} + e^{-i(k(l-j)\pi/N)}) - \sum_{k=1}^{N-1} (e^{i(k(l+j)\pi/N)} + e^{-i(k(l+j)\pi/N)}) \right] \\
&= \frac{1}{2N} \left[\sum_{k=1}^{N-1} [(e^{\frac{i(l-j)\pi}{N}})^k + (e^{\frac{-i(l-j)\pi}{N}})^k] - \sum_{k=1}^{N-1} [(e^{\frac{i(l+j)\pi}{N}})^k + (e^{\frac{-i(l+j)\pi}{N}})^k] \right] \\
&= \frac{1}{2N} \left[\frac{1-(e^{\frac{i(l-j)\pi}{N}})^N}{1-e^{\frac{i(l-j)\pi}{N}}} - 1 + \frac{1-(e^{\frac{-i(l-j)\pi}{N}})^N}{1-e^{\frac{-i(l-j)\pi}{N}}} - 1 \right] - \\
&\quad \frac{1}{2N} \left[\frac{1-(e^{\frac{i(l+j)\pi}{N}})^N}{1-e^{\frac{i(l+j)\pi}{N}}} - 1 + \frac{1-(e^{\frac{-i(l+j)\pi}{N}})^N}{1-e^{\frac{-i(l+j)\pi}{N}}} - 1 \right] \\
&= \frac{1}{2N} \left[\left(\frac{1-e^{i(l-j)\pi}}{1-e^{i\frac{(l-j)\pi}{N}}} + \frac{1-e^{-i(l-j)\pi}}{1-e^{-i\frac{(l-j)\pi}{N}}} \right) - \left(\frac{1-e^{i(l+j)\pi}}{1-e^{i\frac{(l+j)\pi}{N}}} + \frac{1-e^{-i(l+j)\pi}}{1-e^{-i\frac{(l+j)\pi}{N}}} \right) \right] \\
&= \frac{1}{2N} \left[\left(\frac{1-\cos(l-j)\pi}{1-e^{i\frac{(l-j)\pi}{N}}} + \frac{1-\cos(l-j)\pi}{1-e^{-i\frac{(l-j)\pi}{N}}} \right) - \left(\frac{1-e^{i(l+j)\pi}}{1-e^{i\frac{(l+j)\pi}{N}}} + \frac{1-e^{-i(l+j)\pi}}{1-e^{-i\frac{(l+j)\pi}{N}}} \right) \right].
\end{aligned}
$$

   Letting $\theta = (l-j)\pi$, the first term in the brackets above is

$$
\frac{1-\cos\theta}{1-e^{i\frac{\theta}{N}}} + \frac{1-\cos\theta}{1-e^{-i\frac{\theta}{N}}} = (1-\cos\theta) \frac{1-e^{i\frac{\theta}{N}} + 1 - e^{-i\frac{\theta}{N}}}{(1-e^{i\frac{\theta}{N}})(1-e^{-i\frac{\theta}{N}})} = (1-\cos\theta).
$$

   A similar calculation shows that the second term is $(1-\cos(l+j)\pi)$. Putting everything together, for $l \neq j$,

$$
(P^2)_{l,j} = \frac{1}{2N} [(1-\cos(l-j)\pi) - (1-\cos(l+j)\pi)] = \frac{1}{2N} [\cos(l+j)\pi) - \cos(l-j)\pi)] .
$$

   Since l, j are integers, either $l-j$ and $l+j$ are **both** even or they are **both** odd (if $l-j$ is even then $l+j = l-j+(2j)$ is even, etc.). Thus (finally!), $(P^2)_{l,j} = 0$ for $l \neq j$.

If $l = j$ the full derivation above is not valid since once we summed the partial series the denominators in the first term are zero. However, since $1 \leq l, j \leq N-1$ we must have that $l + j < 2N$, thus the denominators in the second term are never zero. So going back a few steps in the derivation above,

$$
\begin{aligned}
(P^2)_{l,j} &= \frac{1}{2N}\left[\sum_{k=1}^{N-1} 2\cos\left(k(l-j)\pi/N\right) - \sum_{k=1}^{N-1} 2\cos\left(k(l+j)\pi/N\right)\right] \\
&= \frac{1}{2N}\left[\sum_{k=1}^{N-1} 2\cos\left(k0\pi/N\right) - \sum_{k=1}^{N-1} 2\cos\left(k(l+j)\pi/N\right)\right] \\
&= \frac{1}{2N}\left[2(N-1) - (1 - \cos\left((l+j)\pi\right) - 2)\right] \\
&= \frac{1}{2N}\left[2(N-1) - (1 - \cos\left(2l\pi\right) - 2)\right] = \frac{1}{2N}\left[2(N-1) - (1 - 1 - 2)\right] = 1
\end{aligned}
$$

Thus, $P^2 = I \quad \Rightarrow \quad P^{-1} = P$. There are most likely more concise ways to show this result.