Math 552	**Scientific Computing II**	**Spring 2020**

Homework Set 2 — Due Friday, 21 February 2020

1. (Integral Mean Value Theorem) Assume the $g \in C[a,b]$ and that f is an integrable function that is either nonnegative or nonpositive throughout the interval $[a,b]$. Then there exists a point $\eta \in [a,b]$ such that

$$\int_a^b g(x)f(x)\,dx = g(\eta)\int_a^b f(x)\,dx\,.$$

2. Suppose that A is $n \times n$ symmetric matrix, i.e. $A^T = A$. A is called *positive definite* if $x^T A\,x > 0$ for all $x \neq 0$ in $\mathbb{R}^n$. Show that the following matrices are positive definite:

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

3. Given a function $f(x)$, use Taylor approximations to show that a 2^{nd} order one-sided approximation to $f'(x_j)$ is given by

$$f'(x_j) \approx \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}\,.$$

Here $f_j = f(x_j)$, $f_{j+1} = f(x_j + h)$, and $f_{j+2} = f(x_j + 2h)$. What is the precise form of the error term? Using the formula approximate $f'(0)$ where $f(x) = e^x$ for $h = 2^(- N)$ for $N = 1{:}12$. Form a table with columns giving h, the approximation, absolute error and absolute error divided by h^2. For each indicate to which values they are converging. Finally, verify that the last column appears to be converging to a value derived using the error term.

4. The method of undetermined coefficients was used to derived the 2^{nd} order centered finite difference approximation to both $f'(x_j)$ and $f''(x_j)$, given respectively by

$$(Df)(x_j) = \frac{f_{j+1} - f_{j-1}}{2h}, \quad (D^2 f)(x_j) = D_-(D_+ f)(x_j) = D_+(D_- f)(x_j) = \frac{f_{j-1} - 2f_j + f_{j+1}}{h^2}$$

Here $f_j = f(x_j)$ and $f_{j\pm1} = f(x_j \pm h)$ where $x_{j\pm1} = x \pm h$. Derive the same approximations as follows:

(a) Find the Lagrange form of the polynomial $P_2(x)$ of degree ≤ 2 such that $P_2(x_j) = f_j$ and $P_2(x_{j\pm1}) = f_{j\pm1}$.

(b) Compute $P_2'(x)$ and show that $P_2'(x_j) = (Df)(x_j)$.

(c) Compute $P_2''(x)$ and show that $P_2''(x_j) = (D^2 f)(x_j)$.

5. (*Method of Undetermined Coefficients*) We derived Simpson's rule to approximate $I(f) = \int_a^b f(x)\,dx$,

$$S = \frac{h}{3}(f(a) + 4f(\frac{a+b}{2}) + f(b)), \qquad h = (b-a)/2\,,$$

by interpolating $f(x)$ at the points $x = a, \frac{(a+b)}{2}, b$, then integrating the interpolant over $[a, b]$. The approximation satisfies

$$I(f) = S - \frac{1}{90} h^5 f^{(4)}(\eta)\,, \quad \text{where } \eta \in (a, b)\,.$$

Note that the term $f^{(4)}(\eta)$ implies Simpson's rule is *exact* if $f(x)$ is a polynomial of degree <= 3, i.e., $P_n(x)$ for $0 \le n \le 3$.

(a) Use the method of undetermined coefficients to derive S. Assume that

$$S = c_1 f(a) + c_2 f(\frac{a+b}{2}) + c_3 f(b)\,,$$

where the coefficients c_1, c_2, and c_3 are to be determined. Evaluate the expression using the *three* functions $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$, and for each compute the exact answer. Then derive and solve a 3×3 linear system for the coefficients.

(b) Now use S to approximate $I(f)$ with $f(x) = x^3$. Is the answer exact? Discuss.

6. Write a MATLAB function M-file **trilu** to find the LU decomposition as discussed in class, $A = LU$, for the tridiagonal $n \times n$ matrix A,

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \beta_2 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \beta_n & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & c_1 & & & \\ & \alpha_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & \alpha_n \end{pmatrix} = LU.$$

The function should output the two n-vectors α and β, and its first line should read:

```
function [alpha,beta] = trilu(a,b,c)
```

Next, write an M-file function **trilu_solve** to solve $Ax = f$, which takes the vectors α, β, c and f and returns x. Its first line should read:

```
function x = trilu_solve(alpha,beta,c,f)
```

Test your code with the 5×5 system with $a_i = 2$, $b_i = -1$, $c_i = -1$, and RHS $f = [1, 0, 0, 0, 1]^T$. The exact solution is clearly $x = [1, 1, 1, 1, 1]^T$. Use MATLAB's **diary** command to save your MATLAB session output showing that your code works properly. Include a copy of both codes.