

1. (A result concerning Lagrange polynomials) Consider a set $\{x_0, x_1, \dots, x_n\}$ of $n+1$ *distinct* points, and the corresponding Lagrange basis functions $\{l_0(x), l_1(x), \dots, l_n(x)\}$. Prove that

$$\sum_{k=0}^n l_k(x) = 1.$$

(Hint - Consider interpolating the function $f(x) = 1$, a polynomial of degree 0, at the points $\{x_0, x_1, \dots, x_n\}$.)

2. Let $f(x) = 1/x$. Take $x_0 = 2$, $x_1 = 3$, $x_2 = 4$.
- (a) Find the Lagrange form and standard form of the interpolating polynomial $P_2(x)$ of $f(x)$ at the given interpolation points. Expand out the Lagrange form to verify that it agrees with the standard form of $P_2(x)$ that you found. Also, verify that $P_2(x_i) = f(x_i)$ for $0 \leq i \leq 2$.

- (b) Use the theorem stated in class to find an upper bound for the error

$$\|f - P_2\|_\infty = \max_{2 \leq x \leq 4} |f(x) - P_2(x)|.$$

- (c) Find $\|f - P_2\|_\infty$ to at least 5 decimal places of accuracy.

3. Approximate the following definite integrals using the composite Trapezoidal rule $T(h)$ for $N = 2, 4, 8, 16, 32$ and 64 , where $h = (b - a)/N$.

- (a) $\int_0^2 3x + 1 \, dx$
(b) $\int_0^2 xe^{-x^2} \, dx$
(c) $\int_0^{2\pi} \cos x + 1 \, dx$

To do so, write an M-file *trap.m*, the first line of which should be

```
function y = trap(f,a,b,N)
```

Include a copy of your code. For each of the functions above make a table, as was done in class, with columns for N , h , $T(h)$, $|\text{error}|$, and $|\text{error}|/h^2$. Are the numbers in the last column converging, and if so, what does it mean? Specifically, comment on the behavior of the error for (a) and (b). If your code is correct, you'll notice that for (c) the last column is not converging, and that the approximation is very accurate. Can you explain why?

4. (Corrected Trapezoidal Rule) Recall that one form of the error term for the composite Trapezoidal rule $T(h)$ is

$$(1) \quad E(h) = -\frac{h^3}{12} \sum_{i=0}^{N-1} f''(\eta_i) = -\frac{h^2}{12} \left(\sum_{i=0}^{N-1} f''(\eta_i) h \right),$$

where $\eta_i \in [x_i, x_{i+1}]$, $x_i = a + ih$, and $h = (b - a)/N$. Note that the last term in the parentheses above in (1) can be viewed as a Riemann Sum approximation of

$$\int_a^b f''(x)dx = f'(b) - f'(a).$$

This suggests we could *correct* (more precisely improve) the accuracy of the Trapezoidal Rule by including this term if the values of $f'(a)$ and $f'(b)$ are available. The resulting numerical integration rule is called the Corrected Trapezoidal Rule, whose composite form is given by

$$CT(h) = \frac{h}{2} (f(x_0) + 2f(x_1) + \dots + 2f(x_{N-1}) + f(x_N)) - \frac{h^2}{12} (f'(b) - f'(a)).$$

It can be shown that the error for $CT(h)$ when approximating $\int_a^b f(x)dx$ is proportional to h^4 , a significant improvement over composite $T(h)$, whose error is proportional to h^2 .

- (a) Write a MATLAB function M-file to compute $CT(h)$, where the first line is of the form:

```
function y = corrected_trap(f,a,b,N,fpa,fpb)
```

Include a copy of your code.

- (b) To numerically verify the order of $CT(h)$ apply your code to approximate

$$I = \int_0^2 x e^{-x^2} dx$$

for $N = 2, 4, 8, 16, 32, 64$. Make a table, as was done in class, with columns for N , h , $CT(h)$, $|\text{error}|$, and $|\text{error}|/h^4$. Are the numbers in the last column converging, and if so, what does it mean?