2^{nd} / 4^{th} order Poisson Solvers

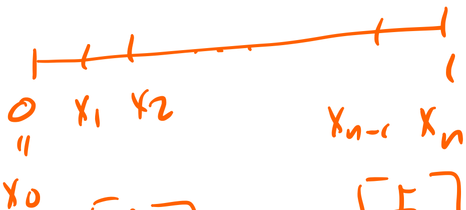
$$-u'' + \sigma u = f$$

$$\boxed{\begin{cases} u'' = f \\ u(0) = u(1) = 0 \end{cases}}$$

Continuous

Choose n

$h = 1/n$, $x_i = i * h$, $0 \le i \le n$



Discrete Problem

$$D^2(u_i) = (u_{i-1} - 2u_i + u_{i+1})/h^2$$

$$\boxed{A_h V_h = F_h}$$

$$V_h = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}, \quad F_h = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

$$A_h = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

d^2/dx^2

(negative of our original A_h)

Eigenstructure of A_h

$$P^{-1} A_h P = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}, \qquad \lambda_k = \frac{-2(1-\cos(k\pi h))}{h^2}$$

$$1 \le k \le n-1$$

$$P_{j,k} = (P_k)_j = \frac{2}{\sqrt{2n}} \sin\left(\frac{kj\pi}{n}\right) \qquad 1 \le j,k \le n-1$$

columns of P form a ORTHONORMAL basis for $\mathbb{R}^{n-1}$

Facts: $\boxed{P^{-1} = P^T = P}$

Orthogonal Bases: $V = \mathbb{R}^n$, $\boxed{S = \underbrace{\{u_1, u_2, \dots, u_n\}}}$

$\boxed{x \in V}$

$\boxed{\text{Orthogonal} \Rightarrow \langle u_i, u_j \rangle = 0 \text{ if } i \neq j}$

$$x = \sum_{i=1}^{n} \alpha_i u_i$$

or

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = x$$

$$\alpha = [x]_S$$

Without knowing that S is an ORTHOGONAL Basis, cost to find α is $O(\frac{2}{3}n^3)$

using orthogonality:

$$\langle x, u_i \rangle = \langle \sum_{j=1}^{n} \alpha_j u_j, u_i \rangle$$ and $\langle \cdot, \cdot \rangle$ is LINEAR

$$= \sum_{j=1}^{n} \alpha_j \langle u_j, u_i \rangle = \alpha_i \langle u_i, u_i \rangle$$

$$\Rightarrow \alpha_i = \frac{\langle x, u_i \rangle}{\langle u_i, u_i \rangle} = \frac{\langle x, u_i \rangle}{\|u_i\|_2^2} = 1$$

If S is ORTHONORMAL

$$\boxed{\alpha_i = \langle x, u_i \rangle}$$

or n inner products

cost $(2n \cdot n) = O(2n^2)$

$$\langle y, y \rangle = \sum_{i=1}^{n} y_i^2 = \|y\|_2^2$$

For us: $V = \mathbb{R}^{n-1}$, $S = \{\underbrace{p_1, p_2, \ldots, p_{n-1}}_{\text{columns of } P}\}$

$x \in V$

$$x = \sum_{i=1}^{n-1} \alpha_i p_i$$

$$\Downarrow$$

$$\begin{bmatrix} \underbrace{p_1 \; p_2 \cdots p_{n-1}} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = x$$

or

$$P\alpha = x$$

or $\alpha = P^{-1} x = \underbrace{Px}$

$$= P^T x$$

$$\alpha = P^T x$$

$$= \begin{bmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_{n-1}^T \end{bmatrix} x = \begin{bmatrix} p_1^T x \\ p_2^T x \\ \vdots \\ p_{n-1}^T x \end{bmatrix}$$

$$= \begin{bmatrix} \langle x, p_1 \rangle \\ \langle x, p_2 \rangle \\ \vdots \\ \langle x, p_{n-1} \rangle \end{bmatrix}$$

$$\Rightarrow \boxed{\alpha_i = \langle x, p_i \rangle}$$

2nd order Solver: $\frac{d^2}{dx^2} = \underbrace{D^2}_{\text{2nd order approx}} + O(h^2)$

$$A_h V_n = F_n$$

$$\Downarrow$$

$$PDP^{-1} V_n = F_n$$

$$\Downarrow$$

$$V_n = \boxed{P}\, \underbrace{D^{-1} \underbrace{P^{-1} F_n}_{\hat{F}_n}}_{\hat{V}_n}$$

$$\hat{V}_n = [V_n]_S$$

$$\boxed{\begin{aligned} P^{-1} A_h P &= D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{bmatrix} \\ A_h &= PDP^{-1} \end{aligned}}$$

$$\boxed{V_n = P D^{-1} P F_n}$$

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & 1/\lambda_{n-1} \end{bmatrix}$$