Math 551 **Scientific Computing I** **Spring 2021**

Homework Set 9 Due 5PM on Wednesday, May 12th

1. Consider the linear system $Ax = b$ where $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. The true answer is $x = [1,\ 1]^T$.

   In class we showed, using the eigen pairs (λ_1, p_1) and (λ_2, p_2) of a given iteration matrix M, that for $x^{(0)} = [0\ 0]^T$ the initial error can be written as

   $$e^{(0)} = \alpha_1 p_1 + \alpha_2 p_2$$

   and for $k \geq 1$

   $$e^{(k)} = \alpha_1 \lambda_1^k p_1 + \alpha_2 \lambda_2^k p_2 \,.$$

   (a) Derive the precise expression for $e^{(0)}$ for both the Jacobi and Gauss-Seidel methods.

   (b) For both methods what is the precise expression for $e^{(k)}$?

   (c) Using your results in (b), prove that both methods will converge.

   (d) Which method converges faster and why?

2. (Optimal Successive Over-Relaxation (SOR)) Given the decompostion $A = L + D + U$ the Gauss-Seidel iteration is written

$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b\,.$$

Introducing the parameter ω, we derived the SOR method

$$Dx^{(k+1)} = Dx^{(k)} - \omega(Lx^{(k+1)} + (D+U)x^{(k)} - b)\,. \tag{1}$$

Note $\omega = 1$ is just the Gauss-Seidel method, and for $\omega \geq 1$ the resulting iteration is called a **Successive Over-Relaxation (SOR)** method.

(a) Solve (1) for $x^{(k+1)}$ to write $x^{(k+1)} = M_\omega x^{(k)} + \tilde{b}$, where M_ω is the SOR iteration matrix. In terms of L, D and U, what is M_ω?

(b) Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Prove that A is symmetric positive-definite.

(c) From the results of D. Young (1950) discussed in class, what is the **exact** expression for the optimal SOR parameter ω for the matrix A?

(d) With the optimal ω from (c), perform 10 iterations of the SOR method with $x^0 = [0, 0, 0, 0]^T$ for the system in (b) and produce the following table:

column 1: k (iteration step)
column 2: $\|e^{(k)}\|_2$ (error norm at step k)
column 3: $\|e^{(k)}\|_2/\|e^{(k-1)}\|_2$ (ratio of successive error norms at step k)

Be sure to include a copy of your code.

(e) To what value does it appear that column 3 is converging and why?