Math 551 **Introduction to Scientific Computing** **Spring 2021**

Homework Set 6 Due Friday, April 2, 2021 9PM

1. Consider the problem $Ax = f$ where A is an $n \times n$ **tridiagonal** matrix,

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{bmatrix}.$$

Note that the right-hand side of the system is denoted by the vector f here. Write out pseudo-code to solve the system assuming that partial pivoting is not implemented. What is the operation count for the forward elimination and the back substitution steps of Gaussian elimination in this case? Count add/sub and mult/div operations separately, but then give the overall order (big O) of the total operations needed. Assume that no pivoting is required.

2. These problems are intended as a verification of a few results we need in the discussion of the LU factorization algorithm, at least in the case of a 3×3 matrices. Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{2,1} & 1 & 0 \\ -m_{3,1} & 0 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{3,2} & 1 \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & 0 & 1 \end{pmatrix}.$$

(b) Show that

$$E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{pmatrix}.$$

(c) Show that $P_1^{-1}{=}P_1$.

3. Find the LU factorization $A = LU$ **by hand** for

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$

At any point in the factorization was pivoting needed?

4. Three common norms for a vector $x \in \mathbb{R}^n$ are the infinity, 1 and 2 norms defined, respectively, by

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i| \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Prove for any $x \in \mathbb{R}^n$ that

$$\|x\|_\infty \le \|x\|_1 \quad \text{and} \quad \|x\|_2 \le \sqrt{n}\|x\|_\infty.$$