Math 551 **Introduction to Scientific Computing** **Spring 2021**

Homework Set 5 Due Friday, March 26th 2021, 9PM

1. Consider Newton's method for finding the $\sqrt{a}$ by finding the positive root of $f(x) = x^2 - a = 0$. Assuming $x_0 > 0$ and $x_0 \neq \sqrt{a}$, show the following:

   (a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

   (b) $x_{n+1}^2 - a = \left(\dfrac{x_n^2 - a}{2x_n}\right)^2$ for $n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n \geq 1$.

   (c) The iterates $\{x_n\}_{n=0}^{\infty}$ are a strictly decreasing sequence for $n \geq 1$. *Hint*: Consider the sign of $x_{n+1} - x_n$.

   (d) A fundamental result concerning the convergence of sequences of real numbers is that if the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and monotonic, then it **converges to a finite limit**. In light of (a)-(c), discuss the convergence of Newton's method for finding $\sqrt{a}$.

2. (Ill-behaved Root Finding) In our analysis of Newton's method we showed that if $f'(\alpha) \neq 0$ (α is a simple, multiplicity 1 root), then second order convergence results provided x_0 is chosen sufficiently close to α. However, if $\boldsymbol{\alpha}$ **is a root of multiplicity** $\boldsymbol{s \geq 2}$ of $f(x)$, then it follows that

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(s-1)}(\alpha) = 0\,.$$

   Then there exists a function $h(x)$ such that

$$f(x) = (x - \alpha)^s h(x)$$

   and $h(\alpha) \neq 0$.

   a) Write out the iteration function $g(x)$ for Newton's method in this case (note: it will involve $h(x)$ and $h'(x)$).

   b) Show that $g'(\alpha) = 1 - 1/s \neq 0$, and explain why this implies **only linear convergence** of Newton's method for a root whose multiplicity is two or greater.

3. The function

$$f(x) = (x - 1)^2 e^x$$

   has a double root at $x = 1$.

   (a) Derive Newton's iteration for this function. Show that the iteration is well-defined so long as $x_n \neq -1$, and that the convergence is similar to that of the bisection method, certainly not quadratic.

   (b) Implement Newton's method and observe its performance starting from $x_0 = 2$.

(c) Could one use the bisection method to find the root? Explain.

4. Consider the following system of nonlinear equations $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

where

$$f(x, y) = 2x^2 - 2xy + 2y^2 - x - y, \qquad g(x, y) = 4x - y + 2.$$

Find an approximate solution to $F(x, y) = 0$ by taking **2** steps of Newton's method with the initial guess $(x_0, y_0)^T = (2, 0)^T$ to find $(x_1, y_1)^T$, and then $(x_2, y_2)^T$. Recall that the inverse of a 2×2 matrix is easily computed by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Write out the complete details **by hand**, with all the arithmetic done by **by hand**.

Newton's method and Fractal Images - **nothing to hand in here!** Take some time to look at the following webpages, which discuss Newton's method in general, and also its *basins of attraction* for computing the roots (some complex) of $x^n - 1 = 0$, the n^{th} roots of unity.

`http://www.personal.psu.edu/faculty/m/x/mxm14/fractal.htm`

and

`http://mathworld.wolfram.com/NewtonsMethod.html`