Homework Set 4 Due Thursday, March 18th

1. Prove for any $x_0 \in \mathbb{R}$ that the iteration $x_{n+1} = g(x_n)$ converges to a unique fix point α where $g(x) = \cos x$. Find the value of α to at least 12 decimal places.

2. Which of the following iterations will converge to the indicated α, provided x_0 is chosen sufficiently close to α. If it does converge, determine the order of convergence.

   (a) $x_{n+1} = \dfrac{15x_n^2 - 24x_n + 13}{4x_n}, \quad \alpha = 1.$

   (b) $x_{n+1} = \dfrac{3}{4}x_n + \dfrac{1}{x_n^3}, \quad \alpha = \sqrt{2}.$

3. Consider the rootfinding problem $f(x) = 0$ with root α, and $f'(x) \neq 0$ for any x.

   (a) Convert it to the fixed-point problem
   $$x = x + cf(x) = g(x)$$
   with c a nonzero constant. How should c be chosen to ensure rapid convergence to α, provided x_0 is chosen sufficiently close to α.

   (b) Apply your method to $x^3 - 5 = 0$ and discuss your results.

4. (Aitken's Extrapolation) Consider the fixed point iteration $x_{n+1} = g(x_n)$. Once the iterates are "*close*" to the root α then $\dfrac{\alpha - x_{n+1}}{\alpha - x_n} \approx g'(\alpha)$ is nearly a constant (using the MVT). Then applying the approximation to the iterates x_n and x_{n-1},
$$\frac{\alpha - x_{n+1}}{\alpha - x_n} \approx \frac{\alpha - x_n}{\alpha - x_{n-1}},$$
or equivalently $(\alpha - x_{n+1})(\alpha - x_{n-1}) \approx (\alpha - x_n)^2$.

   (a) Solve this expression for α, an *extrapolated* value, which in general is a significant improvement in the approximation to the fixed point.

   (b) With $x_0 = 1.8$, and the fixed-point problem
   $$x = g(x) = x - \frac{x^2 - 3}{2}$$
   compute the iterates $\{x_1, \ldots, x_9\}$ as follows

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x_1 = g(x_0)
x_2 = g(x_1)
x_3 = extrapolate using x_0, x_1 and x_2
x_4 = g(x_3)
x_5 = g(x_4)
x_6 = extrapolate using x_3, x_4 and x_5
x_7 = g(x_6)
x_8 = g(x_7)
x_9 = extrapolate using x_6, x_7 and x_8
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Display you results using *format long g*, and compute the error, $|\sqrt{3} - x_9|$.

(c) Now, again with $x_0 = 1.8$ compute a new set of approximations $\{x_1, \ldots, x_9\}$ using $g(x)$ but WITHOUT extrapolation, and compute the error for this new x_9. Is this approximation better or worse than that found in **b** above. Comment.