Math 551 **Introduction to Scientific Computing** **Spring 2021**

Homework Set 3 Due Thursday 9PM, 4th March

1. Given a function $f(x)$, use Taylor approximations to derive a second order approximation to $f'(x_0)$ is given by

$$f'(x_0) = af(x_0 - h) + bf(x_0 + h) + cf(x_0 + 2h) + O(h^2)\,.$$

   What is the precise form of the error term? Using the formula approximate $f'(0)$ where $f(x) = \sin x$ for $h = 1/(2^p)$ for $p = 1:15$. Form a table with columns giving h, the approximation, absolute error and absolute error divided by h^2. For each indicate to which values they are converging. Finally, verify that the last column appears to be converging to a value derived using the error term.

2. The floating point representation of a number is $x = \pm(0.b_1 b_2 \ldots b_n)_\beta \times \beta^e$, where $b_1 \neq 0$, $-M \leq e \leq M$. Suppose $\beta = 2$, $n = 6$, and $M = 4$.
   (a) Find the smallest positive (x_{min}) and largest positive (x_{max}) floating point numbers that can be represented. Give the answers in decimal form (base 10).
   (b) What is the machine epsilon, *eps*, of this number system?
   (c) Find the floating point number in this system that is closest to $x = 2\pi$.

3. Let the initial interval used for the Bisection method is $[\pi, 3\pi]$. What is the minimum number of steps to guaranteed that the approximation is within a tolerance of 10^{-10}.

4. Use the Bisection method to find the following to within a tolerance of 10^{-8}:
   (a) The real root of $x^3 - x^2 - x - 1 = 0$.
   (b) The smallest positive root of $\cos x - \sin x - 1/2 = 0$.
   (c) The smallest positive root of $\tan x - x = 0$.
   (d) The root of $\tan x - x = 0$ closest to $x = 100$.

   Include the output for each and be sure to display the root(s) to full precision.