Math 551 4/8 (look over Chpt. 4)

$Ax=b$: ① $\underbrace{A=LU}_{O(\frac{2}{3}n^3)}$ ② $\underbrace{L(\overbrace{Ux}^{z})=b}_{O(2n^2)}$ Direct Method

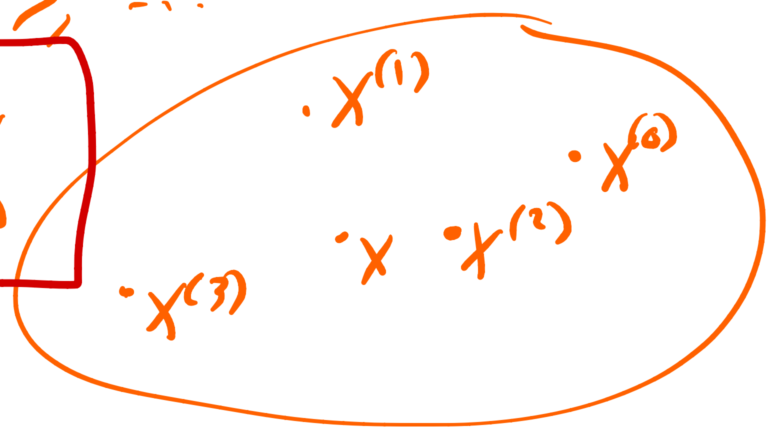
For n "Large" $O(\frac{2}{3}n^3)$ is PROHIBITIVE!

Iterative Method

Guess $x^{(0)}$, and generate $x^{(k)}$

for $k=1,2,3,\ldots$

$$\boxed{\lim_{k\to\infty} x^{(k)} \overset{?}{=} x}$$



Error Analysis

$Ax = b$

x: exact soln $x = A^{-1}b$

$\tilde{x}$: approximate x

e: $x - \tilde{x}$ error

r: $\underbrace{b - A\tilde{x}}_{\text{computable}}$ residual

$(\lambda, x) = (\frac{1}{10^8}, e)$ $\quad Ae = \frac{1}{10^8} e = r$

Note

(1) $Ae = r$

pf: $Ae = A(x - \tilde{x})$
$= Ax - A\tilde{x}$
$= b - A\tilde{x} = r$

(2) $e = 0$ iff $r = 0$

which follows from (1) since A is invertible:

$Ae = r, \quad e = A^{-1} \cdot r$

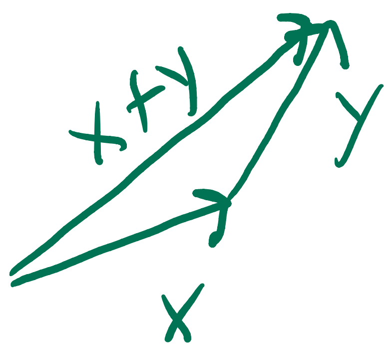
Big Problem: Even if r is "small" e may be "large"

Main Tools

Vector Norm and Matrix Norms

Defn: $x \in \mathbb{R}^n$, a vector norm $\|x\|$ satisfies

1. $\|x\| \ge 0$ and $\|x\| = 0$ iff $x = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$
3. $\|x+y\| \le \|x\| + \|y\|$ $y \in \mathbb{R}^n$



Triangle Inequality

Some Common Norms $\quad x \in \mathbb{R}^n$

p-norm: $\quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad 1 \le p < \infty$

n = 1: $\quad x = [x_1] \qquad \|x\|_p = \left(\sum_{i=1}^{1} |x_i|^p \right)^{1/p} = \left(|x|^p \right)^{1/p} = |x|$ Absolute value

p = 1: $\quad$ 1-norm $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$

p = 2: $\quad \underbrace{\text{2-norm}}_{\text{Euclidean Norm}} \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^T x}$

p → ∞: $\quad$ ∞-norm $\quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$

Ex: $x = [1\ 2\ 3]^T$

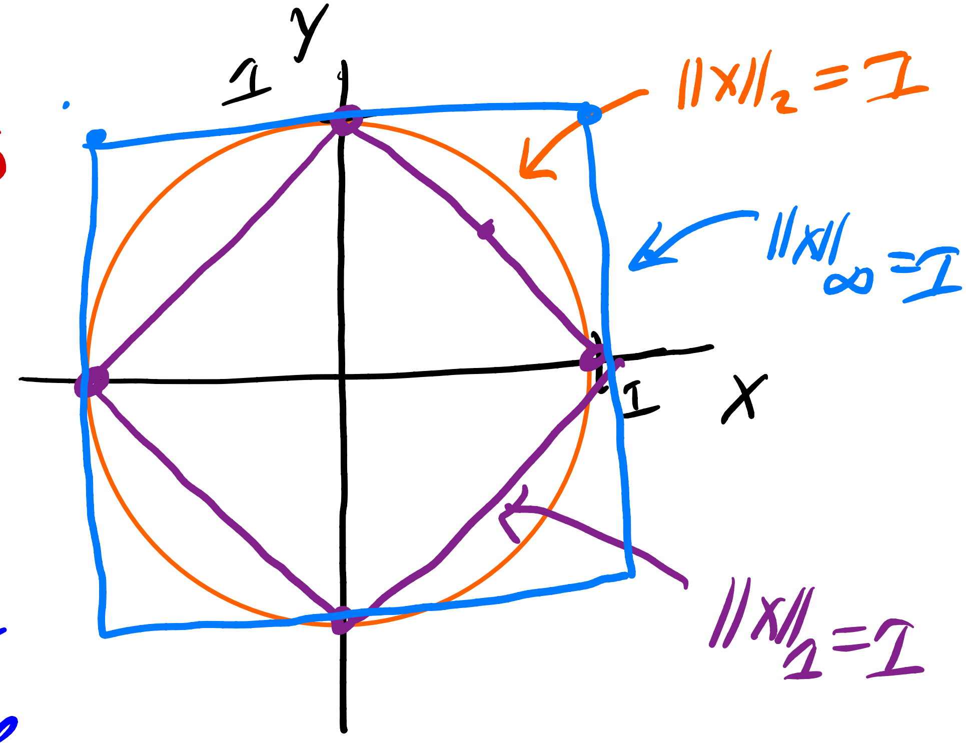
$\|x\|_\infty = \max\{|1|, |2|, |3|\} = 3$

$\|x\|_1 = |1| + |2| + |3| = 6$

$\|x\|_2 = (|1|^2 + |2|^2 + |3|^2)^{1/2} = \sqrt{14}$

Unit Ball: Set of all x such that $\|x\| = 1$

$n=2$: Geometrically comparing norms.



Fact: ALL norms on $\mathbb{R}^n$ are EQUIVALENT in the sense that, if a sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to x in one norm, i.e. $\lim_{k\to\infty} \|x - x^{(k)}\| = 0$, then

$$\lim_{k\to\infty} |||x - x^{(k)}||| = 0.$$

(Math 545)

Matrix Norms

Defn: Given $A, B \in \mathbb{R}^{n \times n}$, a MATRIX norm satisfies

1. $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$
2. $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in \mathbb{R}$
3. $\|A + B\| \leq \|A\| + \|B\|$
4. $\|AB\| \leq \|A\| \|B\|$

Where can we get a matrix norm?

ans: A matrix norm INDUCED by a vector norm.

<u>Defn</u>: Given a vector norm $\|\cdot\|$, the subordinate or <u>INDUCED</u> matrix norm is

$$\|A\| \overset{\text{defn}}{=} \max_{x \neq 0} \frac{\overbrace{\|Ax\|}^{\text{vector norm}}}{\underbrace{\|x\|}_{\text{vector norm}}}$$

$$= \max_{\underbrace{\boxed{\|x\|=1}}_{\text{Closed and Bounded Set}}} \boxed{\|Ax\|} \quad \text{Continous}$$

And we also have

$$(5) \quad \|Ax\| \leq \|A\|\,\|x\|$$