Math 551 4/27

- HW #8 due 4/30 @ 9PM
  Office Hour W 7PM-8PM
- HW #9 assigned

Total of <u>9</u> HWs and <u>3</u> dropped.

Setup: M iteration matrix

Given $x^{(0)}$: $\quad x^{(k+1)} = M x^{(k)} + \tilde{b} \quad k=0,1,\ldots$

Error: $\quad e^{(k)} = M e^{(k-1)}$

Thm: If M is DIAGONIZABLE then

$\lim_{k\to\infty} x^{(k)} = x$ for any $x^{(0)} \iff \rho(M) < 1$

Pf: We prove ($\Longleftarrow$) last class.

($\Longrightarrow$) Suppose $\lim_{k\to\infty} x^{(k)} = x$ for any $x^{(0)}$, and $\rho(M) \geq 1$. We want to reach a contradiction.

If $\rho(M) \geq 1$ then (λ, p) is an e-pair of M where $|\lambda| \geq 1$. Let $\boxed{p \neq 0}$

$p = e^{(0)} = X - X^{(0)}$ or take $X^{(0)} = X - p$, which is some vector. Then

$$e^{(1)} = Me^{(0)} = Mp = \lambda p$$

$$\vdots$$

$$e^{(k)} = \lambda^k p \Rightarrow \|e^{(k)}\| = |\lambda|^k \cdot \underbrace{\|p\|}_{\neq 0}$$

$$\Rightarrow \lim_{k \to \infty} e^{(k)} = \lim_{k \to \infty} |\lambda|^k \|p\| \neq 0$$

Contradiction, so the theorem is proved #

Ex: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Jacobi: $M_J = -D^{-1}(L+U) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

$\rho(M_J) = \frac{1}{2}$ and $\|M_J\|_\infty = 1/2 < 1$

e-pairs: $\Big(\underbrace{\tfrac{1}{2}}_{\lambda_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{p_1}\Big)$ $\Big(\underbrace{-1/2}_{\lambda_2}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{p_2}\Big)$

error: $e^{(0)} = x - x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = p_1 + 0 p_2$
$= p_1$

$$\|e^{(k)}\| = \|M_J^k e^{(0)}\| = \|M_J^k p_1\| = \|\lambda_1^k p_1\| = |\lambda_1|^k \cdot \|p_1\|$$
$$= \left(\tfrac{1}{2}\right)^k \|p_1\| \to 0 \text{ as } k \to \infty$$

<u>G-S</u>: $M_{GS} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$ $\Big(\underbrace{0}_{\lambda_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{P_1}\Big)$ $\Big(\underbrace{\tfrac{1}{4}}_{\lambda_2}, \underbrace{\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}}_{P_2}\Big)$

$\Rightarrow \rho(M_{GS}) = \frac{1}{4}$ while $\|M_{GS}\|_\infty = 1/2$

<u>error</u>: $e^{(0)} = x - x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{P_1} - 2 \underbrace{\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}}_{P_2}$

$$e^{(1)} = M_{GS} e^{(0)} = M_{GS}(3P_1 - 2P_2) = 3 M_{GS} P_1 - 2 M_{GS} P_2$$

$$= 3 \cdot \lambda_1 P_1 - 2 \lambda_2 P_2 = 0 \cdot P_1 - 2\left(\tfrac{1}{4}\right) P_2$$

$$= -2\left(\tfrac{1}{4}\right) P_2$$

$\Rightarrow e^{(k)} = -2\left(\frac{1}{4}\right)^k P_1 \Longrightarrow 0$ as $k \to \infty$

since $\left(\frac{1}{4}\right)^k \to 0$.

$k = 1, 2, \ldots$

SOR (Sucessive Over Relaxation)

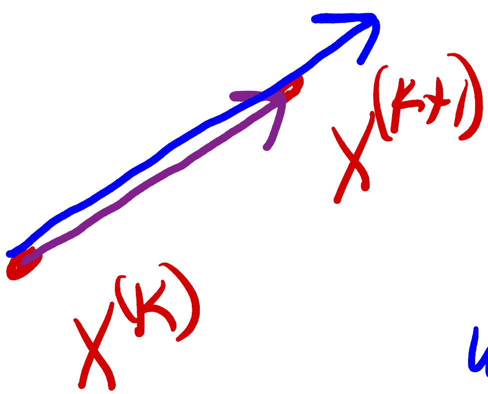
$$A = L + D + U$$

GS: $(L+D)\,x^{(k+1)} = -U x^{(k)} + b$

$$\Rightarrow \quad D x^{(k+1)} = D x^{(k)} - L x^{(k+1)} - D x^{(k)} - U x^{(k)} + b$$

$$D x^{(k+1)} = D x^{(k)} - \left[L x^{(k+1)} + (D+U) x^{(k)} - b\right]$$

$$\Rightarrow \quad \boxed{x^{(k+1)} = x^{(k)} + w \underbrace{\left(-D^{-1}\left[L x^{(k+1)} + (D+U) x^{(k)} - b\right]\right)}_{\text{correction}}}$$



$w = 1 \Rightarrow$ G-S

$w > 1 \Rightarrow$ over relaxation

Let ω be a positive constant

$$x^{(k+1)} = x^{(k)} - \omega\left[D^{-1}\left[Lx^{(k+1)} + (D+U)x^{(k)} - b\right]\right]$$

$$\Rightarrow Dx^{(k+1)} = \underline{Dx^{(k)}} - \underline{\omega}\left[Lx^{(k+1)} + (\underline{D}+U)\underline{x^{(k)}} - b\right]$$

$$\Rightarrow (\omega L+D)\,x^{(k+1)} = ((1-\omega))Dx^{(k)} - \omega Ux^{(k)} + \omega b$$

$$\Rightarrow (\omega L+D)\,x^{(k+1)} = \left[((1-\omega)D - \omega U\right]x^{(k)} + \omega b$$

or $x^{(k+1)} = M_\omega x^{(k)} + \tilde{b}_\omega$

$$\boxed{\begin{gathered}\omega=1\\ M_\omega = -(L+D)^{-1}U = M_{GS}\end{gathered}}$$

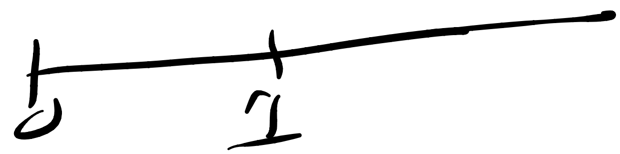
where $M_\omega = (\omega L+D)^{-1}\left[(1-\omega)D - \omega U\right]$

is the SOR iteration matrix!

Ex: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\Rightarrow \omega = 1 \quad \rho(M_\omega) = \rho(M_{GS}) = \frac{1}{4} < 1$$

Question: Can we choose ω so that $\rho(M_\omega) < \rho(M_{GS})$, in fact, as small as possible?



Thm: (D. Young 1950)

(1) If $\rho(M_\omega) < 1 \Rightarrow 0 < \omega < 2$

(2) If A is SYMMETRIC POSITIVE DEFINITE then $\omega^* = \dfrac{2}{1+\sqrt{1-(\rho(M_J))^2}}$ is the OPTIMAL value of ω, $\rho(M_{\omega^*}) \le \rho(M_\omega)$

Defn: $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite if ① $A^T = A$

② $x^T A x > 0$ for $x \neq 0$

Ex: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

① $A^T = A$

② $x^T A x = [x_1 \ x_2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1^2 - 2x_1 x_2 + x_2^2$

$= 2(x_1^2 + x_2^2) - 2x_1 x_2$

$= x_1^2 + x_2^2 + (x_1 - x_2)^2 > 0$ for $x \neq 0$.

So, with this choice of ω^*

$$\rho(M_{\omega^*}) < \rho(M_{GS}) < \rho(M_J) < 1$$

$0.0718 \qquad 1/4 \qquad 1/2$

Back to our example

$$\omega^* = \frac{2}{1+\sqrt{1-[\rho(M_J)]^2}} = \frac{2}{1+\sqrt{1-(1/2)^2}}$$

$$\approx 1.0718$$

and it can be shown that

$$\rho(M_{\omega^*}) = \omega^* - 1 \approx 0.0718$$

Check:

$$M_\omega = \begin{bmatrix} 2 & 0 \\ -\omega & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & \frac{1}{4}\omega^2+1-\omega \end{bmatrix}$$

$\omega = 1$

$$M_{GS} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

K	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	0.5359	0.8231
2	0.9385	0.9798
3	0.9936	0.9980
$\downarrow$	$\downarrow$	$\downarrow$
∞	1	1

$\left.\begin{matrix} k=2 \\ k=3 \end{matrix}\right\} \; \|e^{(3)}\|_\infty < 0.01$