Math 551 4/22

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- HW #7 due today
- HW #8 due 4/30 @ 9pm
- HW #9 due ?

Iterative Methods

$$Ax = b \iff x = Mx + \tilde{b}$$

Given $x^{(0)}$!

$$x^{(k+1)} = Mx^{(k)} + \tilde{b} \quad k = 0, 1, 2 \ldots$$

and

$$\boxed{e^{(k+1)} = Me^{(k)}}$$

So
$$\|e^{(k+1)}\| = \|Me^{(k)}\| = \|MMe^{(k-1)}\|$$
$$= \|M^2 e^{(k-1)}\|$$
$$\vdots$$
$$= \|M^{k+1} e^{(0)}\|$$

or

$$\|e^{(k+1)}\| \le \|M\|^{k+1} \|e^{(0)}\| \Rightarrow$$ If $\|M\| < 1$ then $\lim_{k \to \infty} x^{(k)} = x$ converged!

Recap: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Jacobi: $M_J = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $\|M_J\|_\infty = \frac{1}{2} < 1$

G-S: $M_{GS} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$ and $\|M_{GS}\|_\infty = \frac{1}{2} < 1$

So both converge for any $x^{(0)}$. However, the G-S iteration converged FASTER!

Why?

Eigenvalues (e-vals) and Eigenvector (e-vecs)

$A \in \mathbb{R}^{n \times n}$ (λ, x) is an e-pair if $Ax = \lambda x$, $x \neq 0$

λ → e-val, x → e-vec

How? $Ax = \lambda x$

$\iff Ax - \lambda x = 0$

$\iff \underbrace{(A - \lambda I)x = 0}_{\text{homogeneous system}}$ for $x \neq 0$

$\iff \underbrace{p_A}_{\text{characteristic poly}}(\lambda) = \det(A - \lambda I) = 0$

characteristic poly = poly of exactly degree n $\implies \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are the roots.

Jacobi: $M_J = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

$$\rho_{M_J}(\lambda) = \det(M_J - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{bmatrix}\right) = \lambda^2 - \left(\tfrac{1}{2}\right)^2 = 0$$

$$\Rightarrow \lambda_1 = \tfrac{1}{2}, \quad \lambda_2 = -\tfrac{1}{2}$$

λ_1: $(M_J - \tfrac{1}{2}I)X = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{matrix} -\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 = 0 \\ \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 = 0 \end{matrix}$

or $x_1 = x_2 \Rightarrow P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\left(\tfrac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an e-pair.

λ_2: $(M_J + \tfrac{1}{2}I)X = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 = 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 = 0 \end{matrix}$

or $x_2 = -x_1 \Rightarrow P_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\left(-\tfrac{1}{2}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ is an e-pair

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Note: $S = \{P_1, P_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ IS A BASIS for $\mathbb{R}^2$. ☺

G-S: $M_{GS} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$

$P_{MS}(\lambda) = \det(M_{GS} - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1/2 \\ 0 & \frac{1}{4}-\lambda \end{bmatrix}\right) = \lambda^2 - \frac{1}{4}\lambda$

$= \lambda(\lambda - \frac{1}{4}) = 0 \Rightarrow \lambda_1 = 0$ and $\lambda_2 = 1/4$

$\underline{\lambda_1}$: $(M_{GS} - 0I)X = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \frac{1}{2}x_2 = 0 \\ \frac{1}{4}x_2 = 0 \end{matrix}$

$\Rightarrow X_2 = 0 \Rightarrow P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is an e-pair

$\underline{\lambda_2}$: $(M_{GS} - \frac{1}{4}I)X = \begin{bmatrix} -1/4 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\frac{1}{4}x_1 + \frac{1}{2}x_2 = 0$

$\Rightarrow X_2 = \frac{1}{2}X_1 \Rightarrow P_2 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ or $\left(\frac{1}{4}, \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\right)$ is an e-pair

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<u>Note</u>: $S = \{P_1, P_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\right\}$ is a Basis for $\mathbb{R}^2$ ☺

Defn: $A \in \mathbb{R}^{n \times n}$, then the SPECTRAL RADIUS of A is $\rho(A) = \max\{|\lambda_i| : \lambda_i \text{ is an e-val of } A\}$

M_J: $\|M_J\|_\infty = \frac{1}{2}$ and $\rho(M_J) = \max\{|\frac{1}{2}|, |-\frac{1}{2}|\} = \frac{1}{2} < 1$

M_{GS}: $\|M_{GS}\|_\infty = \frac{1}{2}$ and $\rho(M_{GS}) = \max\{|0|, |\frac{1}{4}|\} = \frac{1}{4} < 1$

We will now show that G-S converges FASTER than Jacobi precisely because

$$0 \leq \rho(M_{GS}) < \rho(M_J) < 1.$$

$$0 \leq 1/4 < 1/2 < 1$$

Defn: $A \in \mathbb{R}^{n \times n}$ is DIAGONALIZABLE if there exists a basis $S = \{p_1, p_2, \dots, p_n\}$ for $\mathbb{R}^n$ comprised SOLELY of e-vecs of A.

So if S is a basis $\Rightarrow$ $P = [p_1\ p_2 \cdots p_n]$ is INVERTIBLE!

Then $AP = A[p_1\ p_2 \cdots p_n] = [Ap_1\ Ap_2 \cdots Ap_n]$

$= [\lambda_1 p_1\ \lambda_2 p_2 \cdots \lambda_n p_n]$ where (λ_i, p_i) are e-pairs

$$= [p_1\ p_2 \cdots p_n] \underbrace{\boxed{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}}}_{D} = PD$$

$\Rightarrow AP = PD$ or $A = PDP^{-1}$

Back to iterative methods

$Mp_i = \lambda_i p_i \quad 1 \le i \le n$

$$e^{(k+1)} = Me^{(k)} = M^2 e^{(k-1)} = \cdots = M^{k+1} e^{(0)}$$

So if M is DIAGONALIZABLE then $S = \{p_1, p_2, \ldots, p_n\}$ is a basis for $\mathbb{R}^n$, so

$$e^{(0)} = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n$$

for unique $\alpha_i \in \mathbb{R}$, $1 \le i \le n$

Then

$$e^{(1)} = Me^{(0)} = M(\alpha_1 p_1 + \cdots + \alpha_n p_n)$$
$$= \alpha_1 M p_1 + \alpha_2 M p_2 + \cdots + \alpha_n M p_n$$
$$= \alpha_1 \lambda_1 p_1 + \alpha_2 \lambda_2 p_2 + \cdots + \alpha_n \lambda_n p_n$$

And since

$$e^{(2)} = Me^{(1)} = \alpha_1 \lambda_1^2 p_1 + \cdots + \alpha_n \lambda_n^2 p_n$$

or

$$e^{(k)} = \alpha_1 \lambda_1^k p_1 + \alpha_2 \lambda_2^k p_2 + \cdots + \alpha_n \lambda_n^k p_n$$

$$\Rightarrow \|e^{(k)}\| = \|\alpha_1 \lambda_1^k p_1 + \cdots + \alpha_n \lambda_n^k p_n\|$$

$$\leq \underbrace{|\alpha_1|}_{\text{fixed}} |\lambda_1|^k \underbrace{\|p_1\|}_{\text{fixed}} + |\alpha_2| |\lambda_2|^k \|p_2\| + \cdots + \underbrace{|\alpha_n|}_{\text{fixed}} |\lambda_n|^k \underbrace{\|p_n\|}_{\text{fixed}}$$

Thm: If M is Diagonalizable, then

$$\lim_{k\to\infty} X^{(k)} = X \text{ for any } X^{(0)} \iff \rho(M) < 1$$

Proof: $\rho(M) < 1 \Rightarrow |\lambda_i| < 1 \quad 1 \le i \le n$

$(\Leftarrow)$ $\rho(M) < 1$

and $\|e^{(k)}\| \le |\alpha_1| \boxed{|\lambda_1|^k} \|p_1\| + \cdots + |\alpha_n| \boxed{|\lambda_n|^k} \|p_n\|$

$\downarrow$ 0 $\qquad$ $\downarrow$ 0

$$\Rightarrow \|e^{(k)}\| \xrightarrow{k\to\infty} 0 \text{ or } \boxed{\lim_{k\to\infty} X^{(k)} = X.}$$

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$S = \{e_1, e_2, \dots, e_n\}$$