Math 551 4/15

- No class on 4/20 (W schedule)
- HW5 returned
- HW7 due 4/21

Recall:

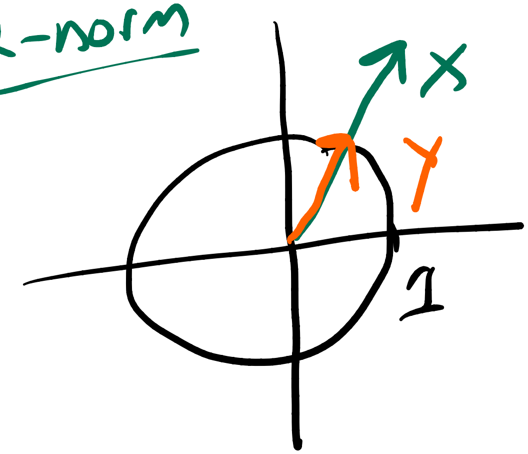
$$\|A\| \overset{\text{defn}}{=} \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

$$= \max_{x\neq 0} \frac{\frac{1}{\|x\|}\,\|Ax\|}{\frac{1}{\|x\|}\,\|x\|}$$

$$= \max_{x\neq 0} \frac{\left\|A\left(\frac{x}{\|x\|}\right)\right\|}{\underbrace{\left\|\frac{x}{\|x\|}\right\|}_{=1}}$$

$$= \max_{\|y\|=1} \frac{\|Ay\|}{1} = \max_{\|x\|=1} \|Ax\|$$

2-norm



$$y = \frac{1}{\|x\|}\cdot x$$

$$\Rightarrow \|y\| = 1$$

Iterative Methods for $Ax=b$

$$A = N - P$$ splitting where N^{-1} exists

$$Ax=b \Rightarrow (N-P)x = b$$

$$\Rightarrow Nx = Px + b$$

$$\Rightarrow x = N^{-1}(Px+b)$$

$$\Rightarrow x = N^{-1}Px + N^{-1}b$$

Choose $x^{(0)}$: $x^{(k+1)} = N^{-1}Px^{(k)} + N^{-1}b$

In Practice $\quad Nx^{(k+1)} = Px^{(k)} + b \quad k=1,2,3,\ldots$

Ex: Jacobi's Method

$$A = L + D + U = \underbrace{D}_{N} + \underbrace{(L+U)}_{-P}$$

Not $LU = A$

$$Ax = b \Rightarrow (L+D+U)x = b$$

$$\Rightarrow Dx = -(L+U)x + b$$

iteration: $D x^{(k+1)} = -(L+U)x^{(k)} + b$

or $x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$

$$x^{(k+1)} = M_J x^{(k)} + D^{-1}b, \quad \underbrace{M_J = -D^{-1}(L+U)}_{\text{Jacobi Iteration Matrix}}$$

Ex: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$L = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $U = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$$\boxed{D x^{(k+1)} = -(L+U)x^{(k)} + b}$$

or
$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$
$$= M_J x^{(k)} + D^{-1}b \qquad \tilde{b}$$

where
$$M_J = -D^{-1}(L+U) = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

Iteration:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$2x_1^{(k+1)} = x_2^{(k)} + 1$$
$$2x_2^{(k+1)} = x_1^{(k)} + 1$$

$$2x_1^{(2)} = x_2^{(1)} + 1 = \frac{1}{2} + 1 = 3/2$$

k	$x_1^{(k)}$	$x_2^{(k)}$	
0	0	0	initial guess
1	1/2	1/2	
2	3/4	3/4	
3	7/8	7/8	
4	15/16	15/16	
$\downarrow$	$\downarrow$	$\downarrow$	
∞	1	1	$\Rightarrow$ Convergence!

$$\|e^{(3)}\|_\infty = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 7/8 \\ 7/8 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1/8 \\ 1/8 \end{bmatrix} \right\|_\infty = \frac{1}{8}$$

Error:

$$Ax = b \qquad x^{(k+1)} = M_J x^{(k)} + \tilde{b}$$

$$x = M_J x + \tilde{b}$$

$$e^{(k)} \overset{\text{defn}}{=} x - x^{(k)}$$

$$= (M_J x + \tilde{b}) - (M_J x^{(k-1)} + \tilde{b})$$

$$= M_J (x - x^{(k-1)})$$

$$e^{(k)} = M_J e^{(k-1)}$$

or

$$e^{(k)} = M_J e^{(k-1)} = M_J (M_J e^{(k-2)}) = M_J^2 e^{(k-2)}$$

$$e^{(k)} = \cdots = M_J^k e^{(0)}$$

$$\|AB\| \le \|A\|\,\|B\|$$

So

$$\|e^{(k)}\| = \|M_J^k e^{(0)}\|$$

$$\|Ax\| \le \|A\|\,\|x\|$$

$$\le \|M_J^k\| \boxed{\|e^{(0)}\|}$$

Fixed vector

$$\|e^{(k)}\| \le \|M_J\|^k \|e^{(0)}\|$$

$$e^{(0)} = x - x^{(0)}$$

So, it is SUFFICIENT to show convergence if we can find an INDUCED matrix norm such that

$$\|M_J\|_\infty = \left\| \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \right\|_\infty = \frac{1}{2} < 1$$

Gauss-Seidel

$A = L + D + U$, $d_{i,i} = a_{i,i} \neq 0$, $1 \le i \le n$

$$Ax = b \Rightarrow (L + D + U)x = b$$
$$\Rightarrow (L+D)x = -Ux + b$$

Given $x^{(0)}$:

$$\boxed{(L+D)x^{(k+1)} = -Ux^{(k)} + b}$$

$$\Rightarrow x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} + (L+D)^{-1}b$$
$$= M_{GS}x^{(k)} + \tilde{b}$$

where $M_{GS} = -(L+D)^{-1}U$ is the Gauss-Seidel Iteration Matrix.

Back to our example: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$M_{GS} = -(L+D)^{-1}U = -\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix} \qquad \|M_{GS}\|_\infty = \frac{1}{2} < \|M_{GS}\|_1 = \frac{3}{4}$$

and $$(L+D)x^{(k+1)} = -Ux^{(k)} + b$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2x_1^{(k+1)} &= x_2^{(k)} + 1 \\ 2x_2^{(k+1)} &= x_1^{(k+1)} + 1 \end{aligned}\right\}$$

Since you have $x_1^{(k+1)}$ use it for any occurence of x_1

Iteration

$$2x_1^{(k+1)} = x_2^{(k)} + 1$$

$$2x_2^{(k+1)} = x_1^{(k+1)} + 1$$

$$2x_1^{(1)} = x_2^{(0)} + 1$$
$$= 0 + 1$$

$$2x_2^{(1)} = x_1^{(1)} + 1$$
$$= \tfrac{1}{2} + 1 = 3/2$$

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	1/2	3/4
2	7/8	15/16
3	31/32	63/64
4	127/128	255/256
$\downarrow$	$\downarrow$	$\downarrow$
∞	1	1

Convergence

Note: For Jacobi $\|e^{(3)}\|_\infty = \frac{1}{8}$

Gauss-Seidel $\|e^{(3)}\|_\infty = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 31/32 \\ 63/64 \end{bmatrix} \right\|_\infty$

$= \left\| \begin{bmatrix} 1/32 \\ 1/64 \end{bmatrix} \right\|_\infty = \frac{1}{32}$

So we see for our $Ax = b$

GS is much faster than Jacobi!!

Why? $\|M_J\|_\infty = \frac{1}{2}$ and $\|M_{GS}\|_\infty = \frac{1}{2}$

They are equal, so WHY is GS faster!