Math 551 4/13

- HW #4 answers
- vector/matrix norms
- Iterative methods $Ax=b$
- HW #7 due 4/21 9PM

Vector Norms $\quad x \in \mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|$$

Induced Matrix Norms $\quad \|\cdot\|, \quad A \in \mathbb{R}^{m\times n}$

$$\|A\| \overset{\text{defn}}{=} \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

(1) $\|A\| \geq 0$, $\quad \|A\| = 0$ iff $A = 0$

⋮

(5) $\|Ax\| \leq \|A\|\|x\|$

Induced 1, 2, ∞ Matrix Norms

1-norm: $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{i,j}| = \max_{1 \le j \le n} \|col_j(A)\|_1$

$= \max$ absolute column sum

∞-norm: $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}| = \max_{1 \le i \le n} \|[row_i(A)]^T\|_1$

$= \max$ absolute row sum

2-norm: $\|A\|_2 = \max_{1 \le i \le n} \{ \sqrt{\lambda_i} : \lambda_i$ is an e-val $A^T A \}$

$= \sqrt{\rho(A^T A)}$

$\rho(B) = \max_{1 \le i \le n} |\lambda_i|$ Spectral Radius of B

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad (A^TA)^T = A^T(A^T)^T = A^TA$

$\|A\|_1 = \max\{4, 6\} = 6$

$\|A\|_\infty = \max\{3, 7\} = 7$

$\|A\|_2 = \sqrt{\rho(A^TA)}$ $\quad$ eig$\left(\begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}\right)$

$= \sqrt{\rho\left(\begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}\right)}$

$\vdots$

$\approx 5.4650.$

Iterative Methods for $Ax=b$

$A = N - P$, splitting where N^{-1} exist

$x_{n+1} = g(x_n)$

$Ax=b \Rightarrow (N-P)x = b$

$\Rightarrow Nx = Px + b$

$\Rightarrow x = N^{-1}Px + \overbrace{N^{-1}b}^{\text{Fixed}}$

Choose $x^{(0)}$: $\boxed{x^{(k+1)} = \underbrace{\boxed{N^{-1}P}}_{M} x^{(k)} + N^{-1}b \quad k=0,1,2,\ldots}$

implemented as $\underbrace{N}_{LU} x^{(k+1)} = Px^{(k)} + b$

Jacobi Method

$$A = L + D + U = \underbrace{D}_{N} + \underbrace{(L+U)}_{-P}$$

$$Ax = b \Rightarrow (L+D+U)x = b$$

$$\Rightarrow Dx = -(L+U)x + b$$

iteration $\quad Dx^{(k+1)} = -(L+U)x^{(k)} + b$

or $\quad x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$

$$= M_J x^{(k)} + \tilde{b}$$

where $\quad M_J = -D^{-1}(L+U)$ Jacobi Iteration Matrix

Ex: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$L = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Iteration: $D x^{(k+1)} = -(L+U) x^{(k)} + b$

or

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1} b$$

$$= M_J x^{(k)} + D^{-1} b$$

$$M_J = -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$