Math 551 — 3/9

- HW #4 due Th 3/18
- HW #3 returned by 3/11
- HW #2 returned by 3/16

Fixed-pt Methods

α is a fixed-pt of $g(x)$, i.e. $g(\alpha) = \alpha$

method:

① choose x_0

② iterate

$$x_{n+1} = g(x_n)$$

$n = 0, 1, 2, \dots$

$$\lim_{n\to\infty} x_n = \alpha \,?$$

Recall:

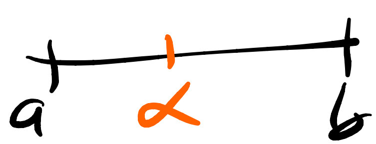
Thm: A1: $g([a,b]) \subseteq [a,b]$

A3: $|g'(x)| \leq \rho < 1$ for $x \in [a,b]$ $\implies$ $g(x)$ has a unique fixed-pt $\alpha \in [a,b]$

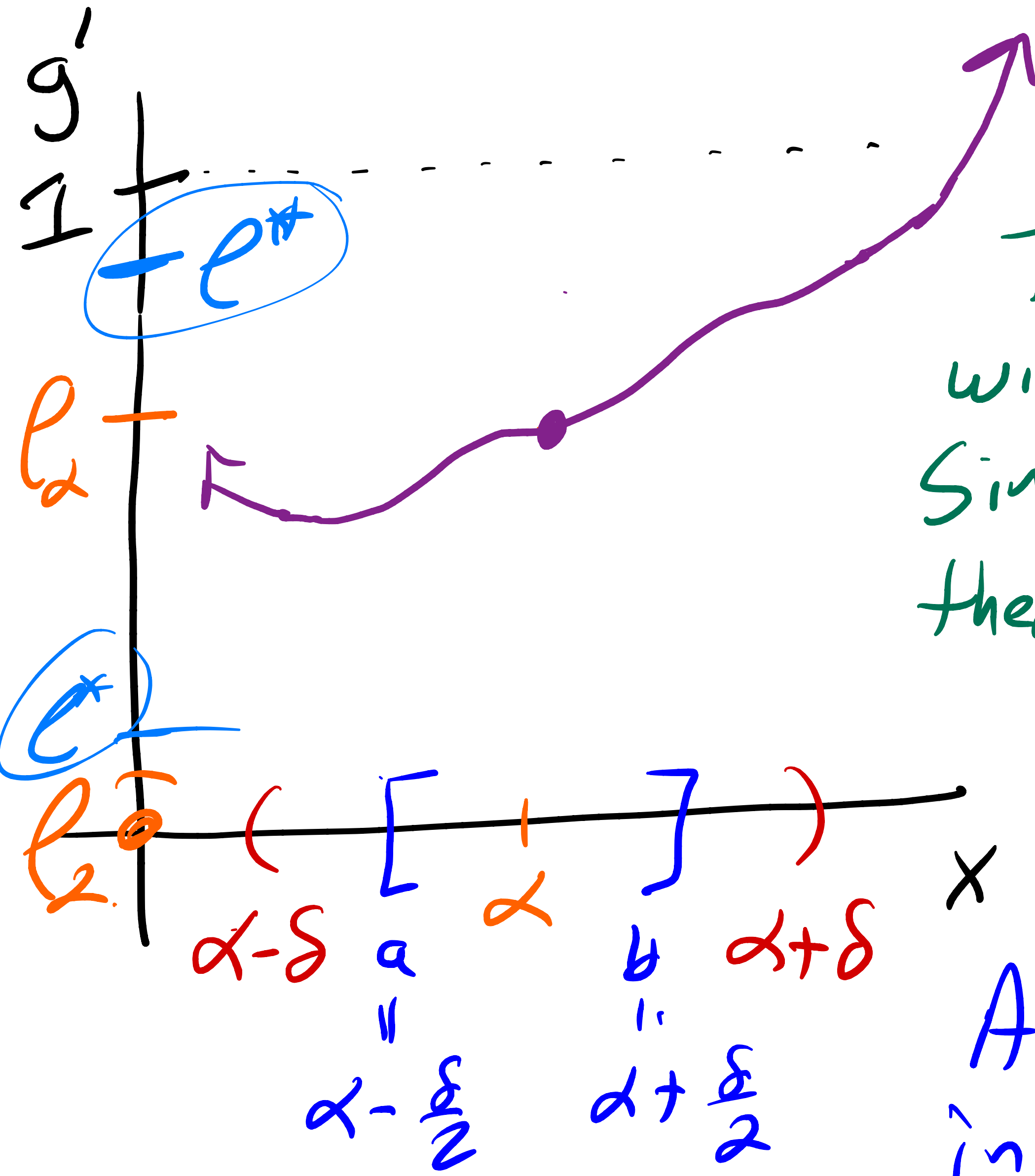
$$\boxed{|\alpha - x_{n+1}| \leq \rho^{n+1} |\alpha - x_0|}$$

Thm: Under the same assumptions if $x_0 \in [a,b]$ $\implies$ iteration $x_{n+1} = g(x_n)$, $n = 0, 1, 2 \ldots$ converges to α.

Note: Both thms assume properties of $g(x)$ on $[a,b]$



In fact, all you need is info about $g(x)$ <u>at</u> α.



Suppose $g \in C^1(\mathbb{R})$ and $|g'(\alpha)| = \rho_\alpha < 1$.

Then there exists ρ^* with $0 < \rho_\alpha < \rho^* < 1$.

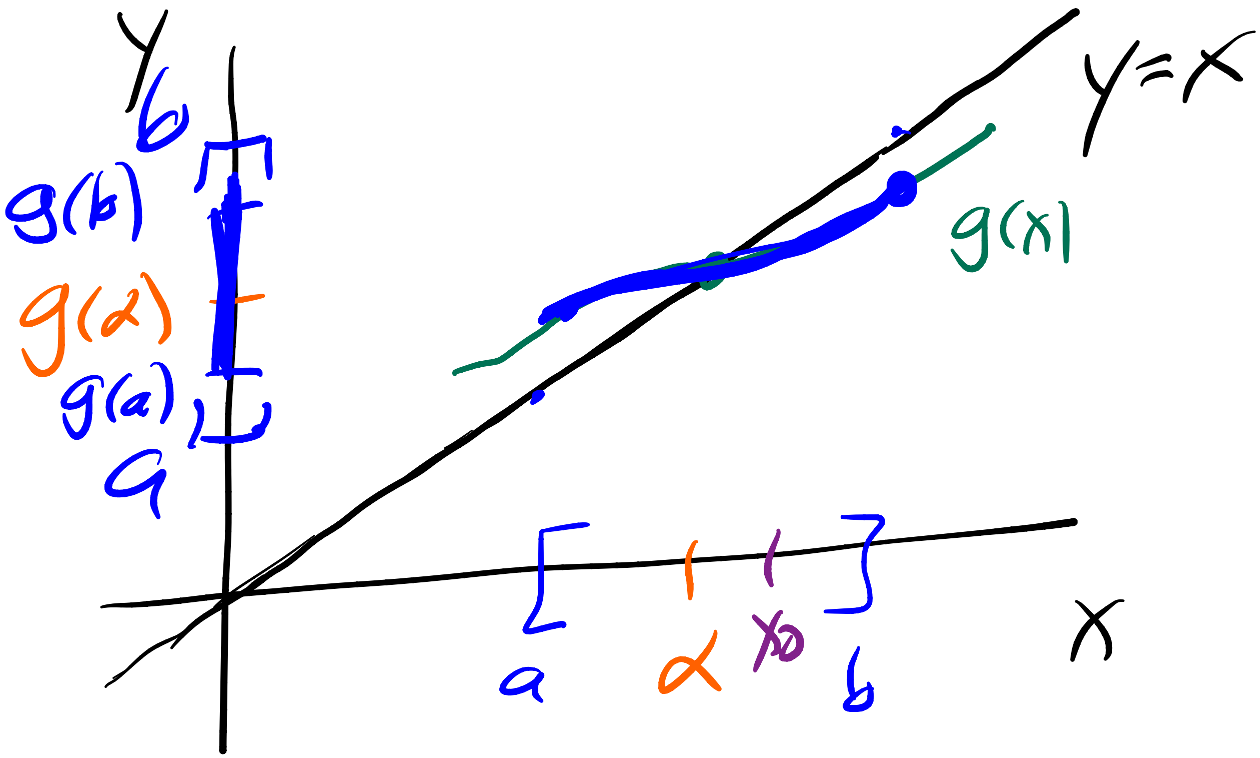
Since $g'(x)$ is continuous, there is a $\delta > 0$ such that

$$|g'(x)| \le \rho^* < 1$$

for $x \in (\alpha-\delta, \alpha+\delta)$.

And $[a,b]$ can be used in our Thms!

Note: We have $g([a,b]) \subseteq [a,b]$



since

$$|g'(x)| \le e^* < 1$$

in $[a,b]$.

Ex: $F(x) = x^2 - 3 = 0$, $\alpha = \sqrt{3}$

$\boxed{g_1(x) = \frac{3}{x}, \ |g_1'(\sqrt{3})| = 1}$ DIVERGES

Newton's Method

$g_3(x) = x - \frac{x^2-3}{2x}$ $\Rightarrow$ $|g_3'(\sqrt{3})| = 0$

Best case possible!

$g_3'(x) = \frac{1}{2}\left(1 - \frac{3}{x^2}\right)$

Note: $g_3''(x) = \frac{1}{2}\left(\frac{6}{x^3}\right)$

So $|g_3''(\sqrt{3})| \neq 0$

Converges! $P=2$

$g_2(x) = x - \frac{x^2-3}{2}$

$|g_2'(\sqrt{3})| \approx 0.7321$

$x_0 = 1.5$

$P=1$

Converges!

Defn (Order of Convergence)

A sequence $\{x_n\}_{n=0}^{\infty}$ converges with ORDER p to α if for $n \geq N > 0$ (eventually!)

$$|\alpha - x_{n+1}| \leq C|\alpha - x_n|^p \quad (p \geq 1)$$

for some $C > 0$. If $p = 1$ the convergence is LINEAR and we require $0 < C < 1$. In this case C is called the RATE of convergence.

Bisection:

We showed

$$|\alpha - x_{n+1}| \leq \frac{1}{2^{n+1}}(b-a)$$

$$|\alpha - x_n| \leq \frac{1}{2^n}(b-a)$$

$$\Longrightarrow \quad |\alpha - x_{n+1}| \leq \frac{1}{2}|\alpha - x_n|$$

So Bisection is LINEAR ($p=1$)

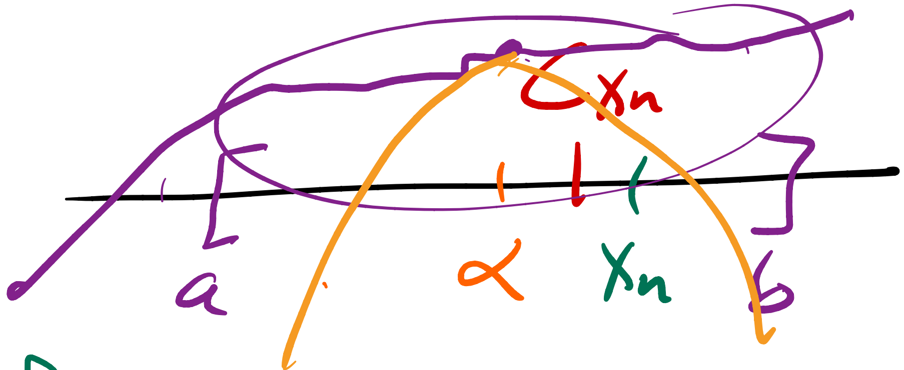
with rate $C = \frac{1}{2}$.

Thm: Suppose $g \in C^p([a,b])$ and $g(\alpha) = \alpha$ where $\alpha \in [a,b]$. If

$$0 = g'(\alpha) = g''(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0$$

but $g^{(p)}(\alpha) \neq 0$ where $p \geq 2$. Then if x_0 is chosen sufficiently close to α, the iteration $x_{n+1} = g(x_n)$ converges to α with order p.

pf:



$g^{(p)}([a,b]) \neq 0$

Taylor expansion of $g(x)$ about α is

$$g(x_n) = g(\alpha) + g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2}(x_n - \alpha)^2 + \cdots + \frac{g^{(p-1)}(\alpha)}{(p-1)!}(x_n - \alpha)^{p-1} + \frac{g^{(p)}(C_{x_n})}{p!}(x_n - \alpha)^p$$

So

$$\underbrace{g(x_n)}_{x_{n+1}} = \underbrace{g(\alpha)}_{\alpha} + \frac{g^{(p)}(c_{x_n})}{p!}(x_n - \alpha)^p$$

So

$$x_{n+1} - \alpha = \frac{g^{(p)}(c_{x_n})}{p!}(x_n - \alpha)^p$$

Let $C = \max_{x \in [a,b]} |g^{(p)}(x)| > 0$

So $|x_{n+1} - \alpha| \le C |x_n - \alpha|^p$ #