Math 551 3/4

HW #2 due 6PM HW2_5763.pdf

HW #3 due 9PM HW3_5763.pdf

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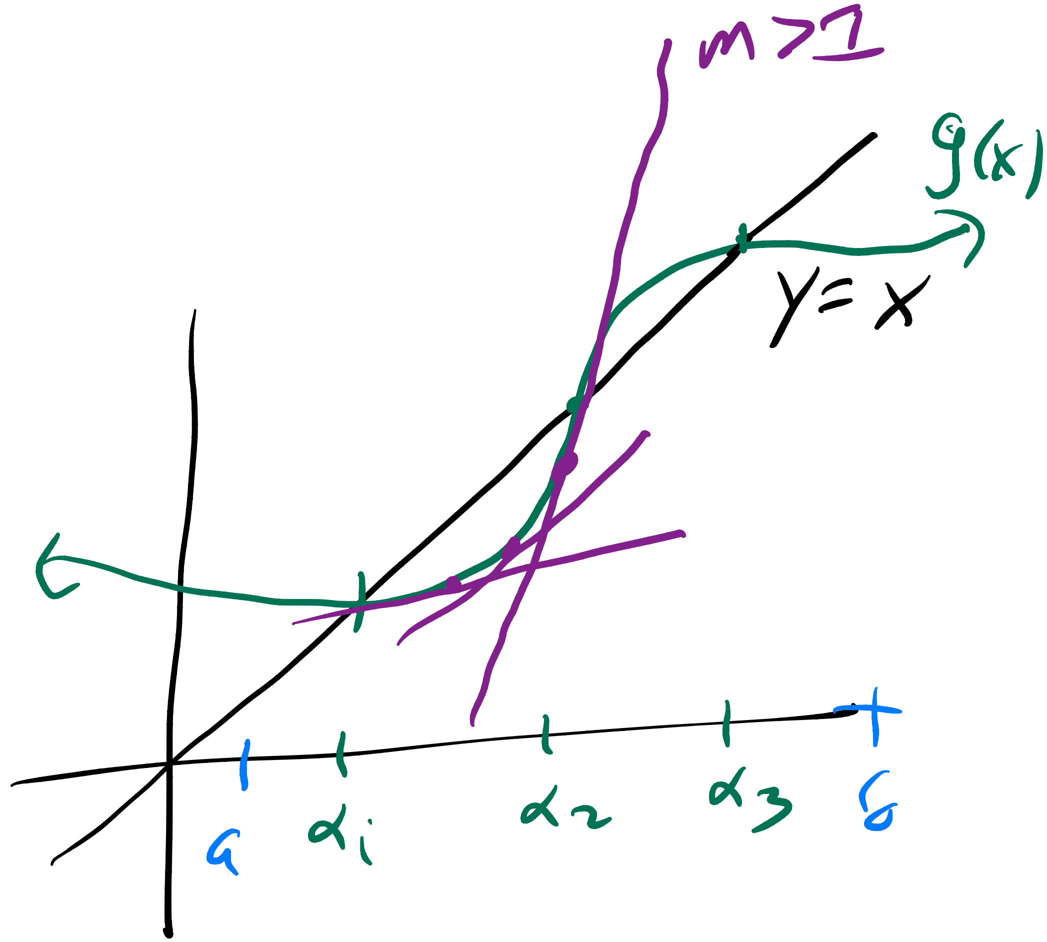
Fixed-Pt Methods

given $g(x)$
Find fixed-pts of g!



Iteration

① Choose x_0 (initial guess)

② Generate $x_1, x_2, \ldots$

$$x_{n+1} = g(x_n)$$

$n = 0, 1, 2, \ldots$

Q: $\lim_{n \to \infty} x_n = \alpha$?

<u>Thm</u>: (Existence of Fixed-Pts)

Given $g(x)$ and $[a,b]$

<u>A1</u>: $g(x)$ maps $[a,b]$ into $[a,b]$

or $g([a,b]) \subseteq [a,b]$

<u>A2</u>: $g \in C([a,b])$

<u>A3</u>: There is a ℓ such that

$|g'(x)| \leq \ell < 1$ for $x \in [a,b]$.

(Note: A3 $\Rightarrow$ A2 since Diff'ble $\Rightarrow$ Continuity

Then,

(1) A1 + A2 $\Rightarrow$ $g(x)$ has a fixed-pt α in $[a, b]$

(2) A1 + A3 $\Rightarrow$ α is UNIQUE!

Pf: Suppose $g(a) = a$ and/or $g(b) = b$ then we are done, since a and/or b are fixed-pts. If not, then

$$g(a) > a \quad \text{and} \quad g(b) < b.$$

Let $h(x) = x - g(x) \Rightarrow h \in C([a,b])$,

$h(a) = a - g(a) < 0$ $\xRightarrow{\text{IVT}}$ There exists $\alpha \in (a,b)$ such that $h(\alpha) = 0$

$h(b) = b - g(b) > 0$

So $0 = h(\alpha) = \alpha - g(\alpha)$

$\Rightarrow \alpha = g(\alpha)$ so we have found a fixed-pt α of $g(x)$ in $[a,b]$

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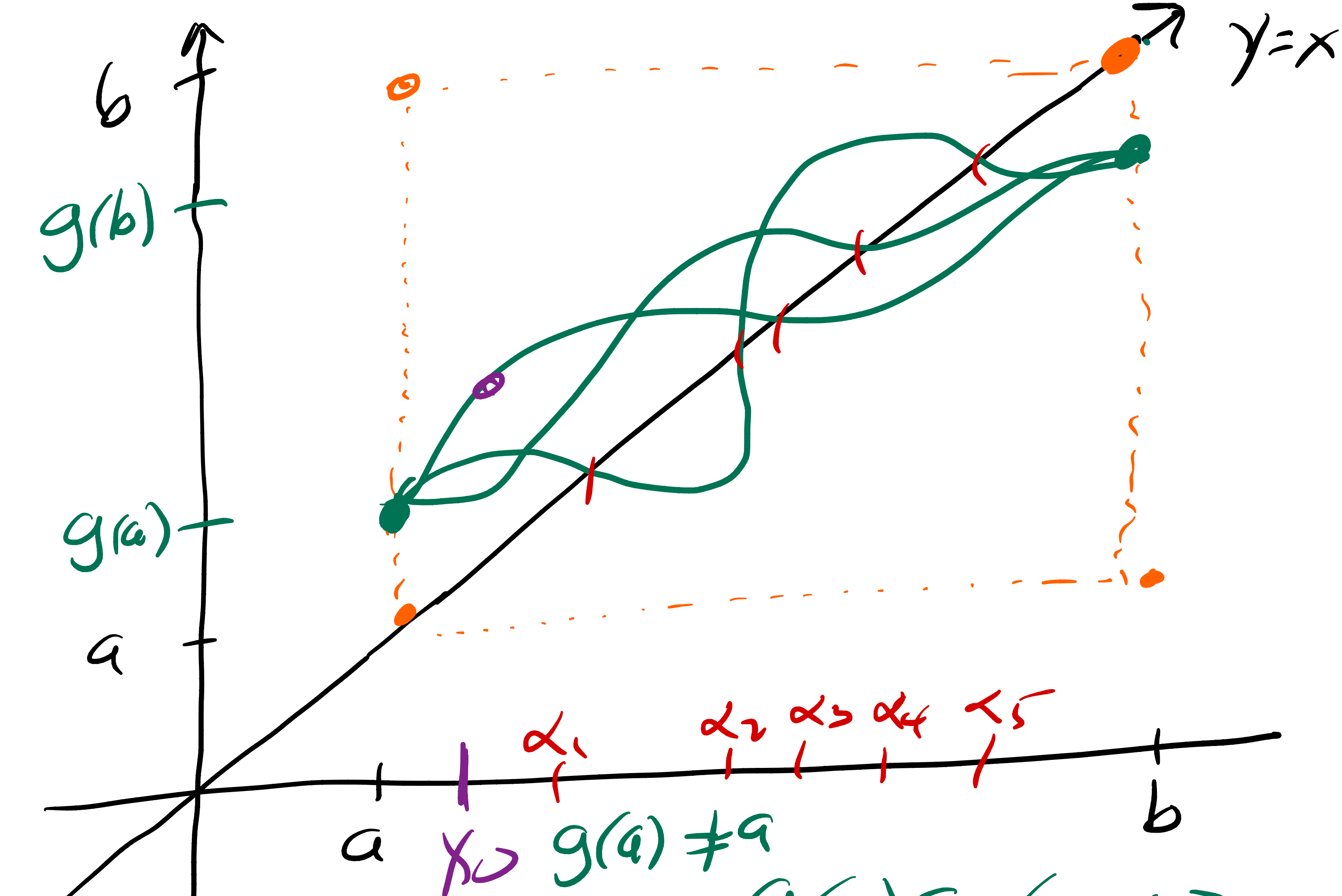

$$g(a) \in (a,b]$$

$$g([a,b]) \subseteq [a,b]$$

(2) Since A3 $\Rightarrow$ A2 then we know there is at least one fixed-pt α in $[a,b]$. We want to show that it is unique. So suppose there are 2 fixed-pts α_1 and α_2, i.e.

$$\alpha_1 = g(\alpha_1)$$
$$\alpha_2 = g(\alpha_2)$$

To show uniqueness we need to show that

$$|\alpha_1 - \alpha_2| = 0$$

We have

$$|\alpha_1 - \alpha_2| = |g(\alpha_1) - g(\alpha_2)|$$

$$\overset{MVT}{=} |g'(c)(\alpha_1 - \alpha_2)|$$

$$= |g'(c)|\,|\alpha_1 - \alpha_2|$$

So

$$|\alpha_1 - \alpha_2| \le \rho\,|\alpha_1 - \alpha_2|$$

or

$$\boxed{\underbrace{(1-\rho)}_{\text{positive}}}\ \boxed{\underbrace{|\alpha_1 - \alpha_2|}_{\ge 0}} \le 0 \implies |\alpha_1 - \alpha_2| = 0 \text{ or } \alpha_1 = \alpha_2$$

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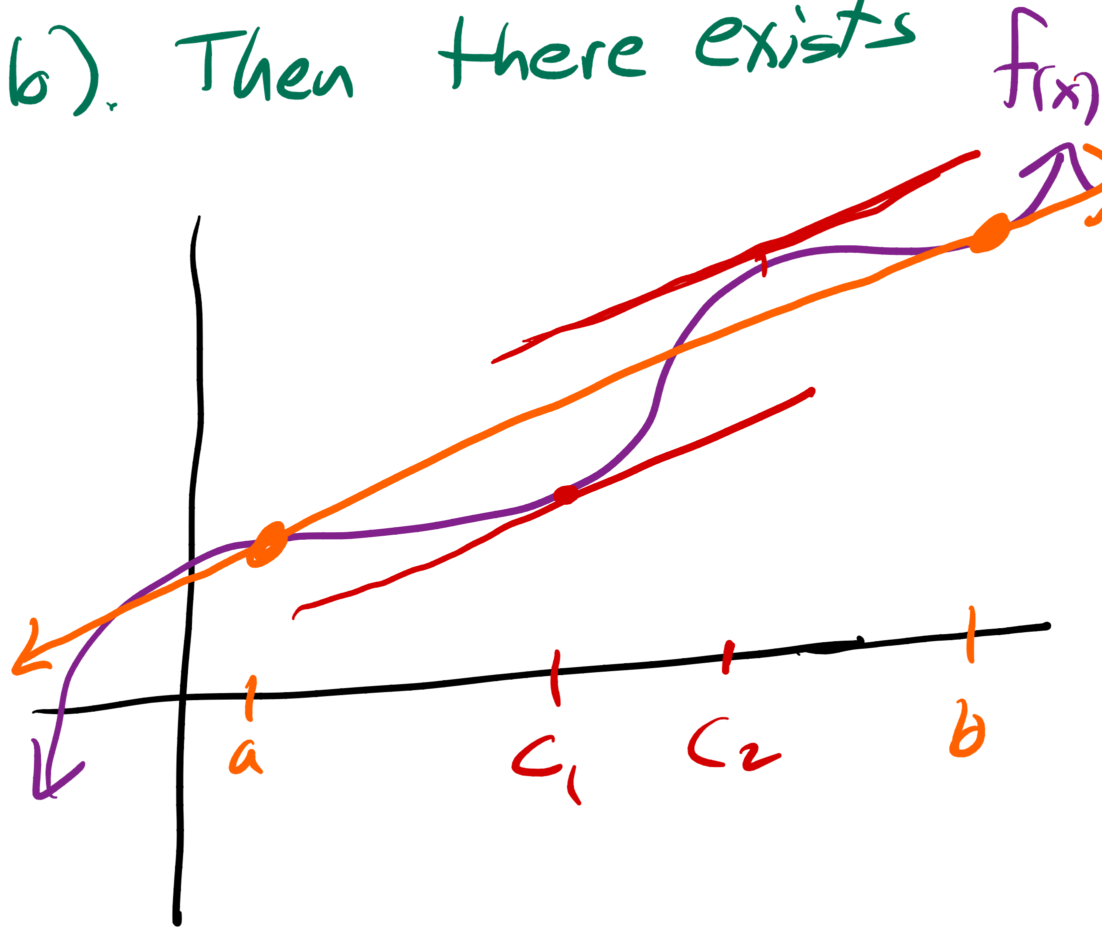
$$0 \le \rho < 1 \implies 1 - \rho > 0$$

Thm Mean Value Theorem (MVT)

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or

$$f(b) - f(a) = f'(c)(b-a)$$

<u>Thm</u> (Convergence of Fixed-Pt methods)
If A1 + A3 hold, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = g(x_n)$, with $x_0 \in [a,b]$, <u>CONVERGES</u> to the <u>unique</u> fixed-point α.

<u>pf</u>: Let $x_0 \in [a,b]$. Then

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)|$$
$$\overset{MVT}{=} |g'(c)|\,|\alpha - x_n|$$

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)|$$

$$\overset{MVT}{=} |g'(c)|\,|\alpha - x_n|$$

$$\overset{A3}{\leq} \rho\,|\alpha - x_n|$$

$$= \rho\,|g(\alpha) - g(x_{n-1})|$$

$$\overset{MVT}{=} \rho\,|g'(c)|\,|\alpha - x_{n-1}|$$

$$\overset{A3}{\leq} \rho^2\,|\alpha - x_{n-1}|$$

$$\vdots$$

SO

$$\boxed{|\alpha - x_{n+1}| \leq \rho^{n+1}\,|\alpha - x_0|}$$

So

$$\lim_{n\to\infty} |\alpha - x_{n+1}| \le \lim_{n\to\infty} \ell^{n+1} |\alpha - x_0|$$

$$= |\alpha - x_0| \lim_{n\to\infty} \ell^{n+1} \qquad 0 \le \ell < 1$$

$$= |\alpha - x_0| \cdot 0$$

$$= 0$$

i.e. $\displaystyle \lim_{n\to\infty} x_n = \alpha.$

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