

$Ax = b$  by GE where  $A$  is invertible.  
 We have assumed that  $a_{k,k} \neq 0$ .

LU:  $U = E_{n-1} E_{n-2} \dots E_3 E_2 E_1 A$  where

upper triangular

$$E_k^{-1} = \begin{bmatrix} I & & & & \\ & I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix}$$

①  $E_k$  is unit lower triangular

⇒

②  $E_k^{-1}$  exists and is TRIVIAL to write it down. ☺

$$\Rightarrow A = (E_{n-1} E_{n-2} \cdots E_2 E_1)^{-1} U$$

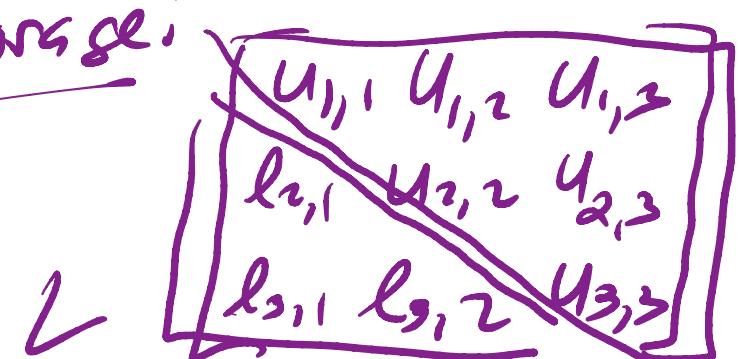
$$= \underbrace{E_1^{-1} E_2^{-1} \cdots E_{n-2}^{-1} E_{n-1}^{-1}}_L U \quad E_1^{-1} E_2^{-1} E_3^{-1}$$

where  $L = \begin{bmatrix} I & & & \\ m_{2,1} & I & & \\ m_{3,1} & m_{3,2} & \ddots & \\ \vdots & \ddots & \ddots & \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n} I \end{bmatrix}$ , a unit lower triangular matrix.

we get it for FREE!

$$\Rightarrow A = L U$$

Storage?

$L$  

$$n=3! \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{2,1} & 1 & 0 \\ -m_{3,1} & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{3,2} & 1 \end{bmatrix}$$

$$\Rightarrow E_1^{-1} E_2^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & 0 & 1 \end{bmatrix}}_{\text{I}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{3,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{bmatrix}$$

I

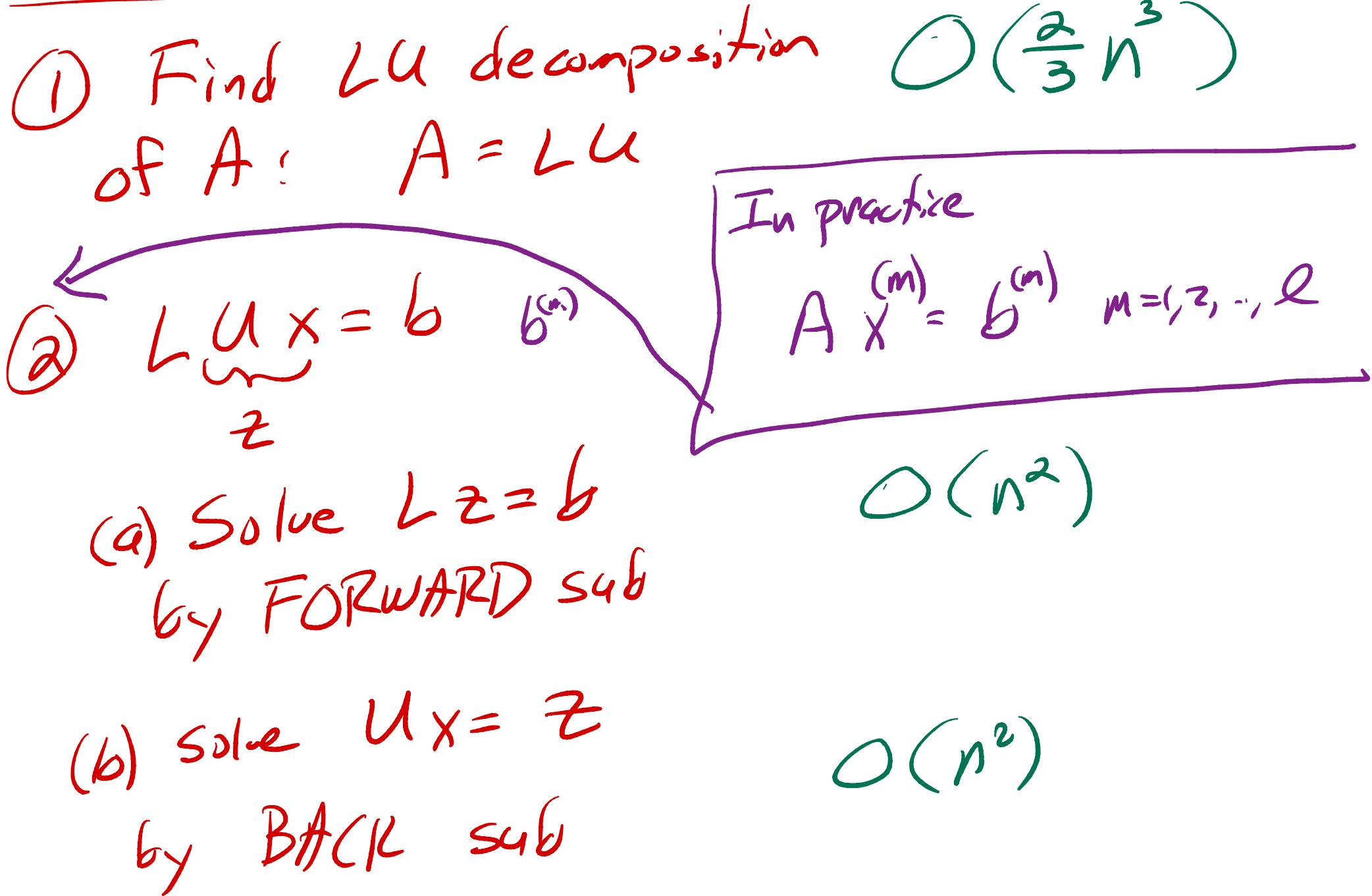
$\Downarrow$

$R_2 + m_{2,1} R_1 \rightarrow R_2$

$R_3 + m_{3,1} R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{bmatrix}$$

Solve  $Ax = b$  (Direct Method)



$$\text{Ex: } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$m_{2,1} = a_{2,1} / a_{1,1} = 2$$

$$m_{3,1} = a_{3,1} / a_{1,1} = 1$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow E_1 A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \cancel{2} & 1 \\ 0 & \cancel{6} & 1 \end{bmatrix} \quad m_{3,2} = \frac{a_{3,2}}{a_{2,2}} = 2$$

$\# \neq 0$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow E_2(E_1 A) = \boxed{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}} = U$$

$$\textcircled{1} \quad A = \underbrace{\tilde{E}_1 \tilde{E}_2^{-1}}_L U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$(Ax = L\tilde{U}x = Lz)$$

$$\textcircled{2} \quad \text{(a) Solve } Lz = b$$

$$\Rightarrow z = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{(b) Solve } Ux = z$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} -1/2 \\ -1 \\ 2 \end{bmatrix}$$

What if at some stage  $a_{k,k} = 0$ ?

answer: If so, then there must be a non-zero entry among  $\{a_{k+1,k}, a_{k+2,k}, \dots, a_{n,k}\}$  otherwise  $A$  would be singular.

Since we assume  $A^{-1}$  exists, then let  $m > k$  be such that  $a_{m,k} \neq 0$  and

perform  $R_k \leftrightarrow R_m$  so  $a_{k,k} \neq 0$ .

However, even if  $a_{k,k} \neq 0$  we may still interchange rows to avoid catastrophic roundoff error.

PARTIAL PIVOTING

## Need to Pivot (partial or full)

Ex:  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

our LU fails!

$$R_1 \leftrightarrow R_2$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

our LU succeeds!

However, even if  $a_{k,k} \neq 0$  we may still want to interchange rows. Why?

$$\left[ \begin{array}{ccc} a_{1,1}^{(1)} & a_{1,2}^{(1)} & a_{1,3}^{(1)} \\ 0 & a_{2,2}^{(2)} & a_{2,3}^{(2)} \\ 0 & a_{3,2}^{(2)} & a_{3,3}^{(2)} \end{array} \right]$$

roundoff  
error

$$m = a_{3,2}^{(2)} / a_{2,2}^{(2)} = 10^{12}$$

$$R_3 - m R_2 \rightarrow R_3$$

$$\begin{aligned} a_{3,3}^{(3)} &= a_{3,3}^{(2)} - m a_{2,3}^{(2)} \\ &= a_{3,3}^{(2)} - m (a_{2,3}^{\text{true}} + \text{error}) \\ &= a_{3,3}^{(2)} - m a_{2,3}^{\text{true}} \end{aligned}$$

m · error

amplifies  
the error

$$\text{If } |a_{2,2}^{(2)}| \ll |a_{3,2}^{(2)}| \Rightarrow |m| \gg 1$$

∴

Fix?: Just before the  $K^{th}$  stage of GE  
choose  $j$  such that

$$|a_{j,K}^{(k)}| = \max \{ |a_{K,K}^{(k)}|, |a_{K+1,K}^{(k)}|, \dots, |a_{n,K}^{(k)}| \}$$

to since  $A^{-1}$  exists!

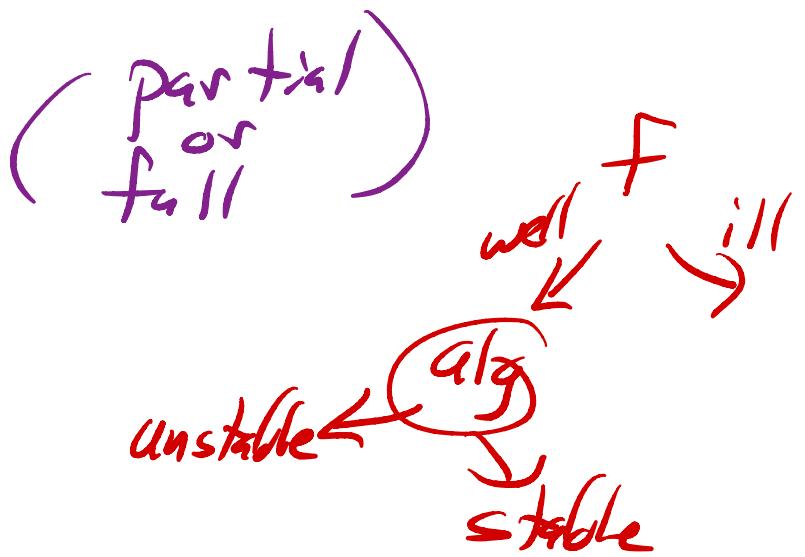
$$A^{(k)} = \begin{bmatrix} a_{1,1}^{(k)} & & & \\ \vdots & \ddots & & \\ 0 & \left[ \begin{array}{c|cc} a_{K,K}^{(k)} & & \\ \hline a_{K+1,K}^{(k)} & & \\ \vdots & & \\ a_{n,K}^{(k)} & a_{n,K+1}^{(k)} & \cdots a_{n,n}^{(k)} \end{array} \right] & & \\ & & \vdots & \\ & & & a_{n,n}^{(k)} \end{bmatrix}$$

Partial                      Full

①  $R_j \leftrightarrow R_K$   
so  
 $|m_{j,K}| \geq 1$   
 $k+1 \leq j \leq n$

## Partial Pivoting:

With PIVOTING (partial or full)  
GE is UNSTABLE.



Pivoting STABILIZES

GE! ☺

$$\frac{\sqrt{8x_1} - \sqrt{x}}{\sqrt{kx_1} + \sqrt{x}} = \frac{1}{\sqrt{kx_1} + \sqrt{x}}$$