Math 551 3/30

$Ax = b$ by GE where A is invertible.

We have assumed that $a_{k,k} \neq 0$.

<u>LU</u>: $\underbrace{U}_{\text{upper triangular}} = E_{n-1} E_{n-2} \cdots E_3 E_2 E_1 A$ where

$$E_k^{-1} = \begin{bmatrix} I & & & & 0 \\ & \ddots & & & \\ & & I & & \\ & & +m_{k+1,k} & \ddots & \\ & 0 & +m_{k+2,k} & & \\ & & \vdots & & \\ & & +m_{n,k} & & I \end{bmatrix}$$

① E_k is unit lower triangular

$\Rightarrow$

② E_k^{-1} exists and is TRIVIAL to write it down. ☺

$$\Rightarrow \quad A = (E_{n-1} E_{n-2} \cdots E_2 E_1)^{-1} U$$

$$= \underbrace{E_1^{-1} E_2^{-1} \cdots E_{n-2}^{-1} E_{n-1}^{-1}}_{L} \; U$$

$E_1^{-1} E_2^{-1} E_3^{-1}$

where $L = \begin{bmatrix} 1 & & & & 0 \\ m_{2,1} & 1 & & & \\ m_{3,1} & m_{3,2} & \ddots & & \\ \vdots & & \ddots & \ddots & \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$, a unit lower triangular matrix.

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$$\Rightarrow \quad A = LU$$

storage?

$$\begin{array}{cc} & U \\ L & \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ l_{2,1} & u_{2,2} & u_{2,3} \\ l_{3,1} & l_{3,2} & u_{3,3} \end{bmatrix} \end{array}$$

$\underline{n=3!}$ $\quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{2,1} & 1 & 0 \\ -m_{3,1} & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{3,2} & 1 \end{bmatrix}$

$$\Rightarrow E_1^{-1} E_2^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & 0 & 1 \end{bmatrix}}_{\mathrm{I}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{3,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{bmatrix}$$

$\Downarrow$

$R_2 + m_{2,1} R_1 \to R_2$

$R_3 + m_{3,1} R_1 \to R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{bmatrix}$$

Solve $Ax = b$ (Direct Method)

① Find LU decomposition of A: $A = LU$ $\quad O\left(\frac{2}{3}n^3\right)$

In practice

$$A x^{(m)} = b^{(m)} \quad m = 1, 2, \ldots, \ell$$

② $L\underbrace{Ux}_{z} = b \quad b^{(m)}$

(a) Solve $Lz = b$ by FORWARD sub $\quad O(n^2)$

(b) Solve $Ux = z$ by BACK sub $\quad O(n^2)$

Ex: $A = \begin{bmatrix} \boxed{2} & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

$$m_{2,1} = a_{2,1}/a_{1,1} = 2$$

$$m_{3,1} = a_{3,1}/a_{1,1} = 1$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow E_1 A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \boxed{2} & 1 \\ 0 & \# & 6 \end{bmatrix} \quad m_{3,2} = \frac{a_{3,2}}{a_{2,2}} = 2$$

$$\# \neq 0$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow E_2(E_1 A) = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

① $A = \underbrace{E_1^{-1} E_2^{-1}}_{L} U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U}$

$(Ax = L\underbrace{Ux}_{z} = Lz)$

② (a) Solve $Lz = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow z = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

(b) Solve $Ux = z$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\Rightarrow x = \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$

What if at some stage $a_{k,k} = 0$?

answer: If so, then there must be a non-zero entry among $\{a_{k+1,k}, a_{k+2,k}, \ldots, a_{n,k}\}$ otherwise A would be singular.

Since we assume A^{-1} exists, then let $m > k$ be such that $a_{m,k} \neq 0$ and perform $R_k \leftrightarrow R_m$ so $a_{k,k} \neq 0$.

However, even if $a_{k,k} \neq 0$ we may still interchange rows to avoid catastrophic roundoff error.

PARTIAL PIVOTING

Need to Pivot (partial or full)

Ex: $A = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{our LU fails!}}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$R_1 \longleftrightarrow R_2$

$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{our LU succeeds!}}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

However, even if $a_{k,k} \neq 0$ we may still want to interchange rows. Why?

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & a_{1,3}^{(1)} \\ 0 & a_{2,2}^{(2)} = 10^{-12} & a_{2,3}^{(2)} \\ 0 & \cancel{a_{3,2}^{(2)}}\ 1 & a_{3,3}^{(2)} \end{bmatrix}$$

roundoff error

$m = a_{3,2}^{(2)} / a_{2,2}^{(2)} = 10^{12}$

$R_3 - m R_2 \rightarrow R_3$

$$a_{3,3}^{(3)} = a_{3,3}^{(2)} - m\, a_{2,3}^{(2)}$$

$$= a_{3,3}^{(2)} - m \left(a_{2,3}^{true} + \text{error} \right)$$

$$= a_{3,3}^{(2)} - m\, a_{2,3}^{true} - m \cdot \text{error}$$

amplifies the error!

If $|a_{2,2}^{(2)}| \ll |a_{3,2}^{(2)}| \Rightarrow |m| \gg 1$:-(

Fix?: Just before the k^{th} stage of GE choose i such that

$$\boxed{|a_{i,k}^{(k)}| = \max\{|a_{k,k}^{(k)}|, |a_{k+1,k}^{(k)}|, \dots, |a_{n,k}^{(k)}|\}}$$

$\neq 0$ since A^{-1} exists!

$$A^{(k)} = \begin{bmatrix} a_{1,1}^{(k)} & \cdots & & \cdots \\ & \ddots & & \\ & a_{k,k}^{(k)} & \cdots & \\ 0 & a_{k+1,k}^{(k)} & & a_{i,j}^{(k)} \\ & \vdots & & \\ & a_{n,k}^{(k)} & a_{n,k+1}^{(k)} \cdots & a_{n,n}^{(k)} \end{bmatrix} \Longrightarrow$$

Partial Full

① $R_i \leftrightarrow R_k$

so

$|m_{j,k}| \le 1$

$k+1 \le j \le n$

Partial Pivoting:

With PIVOTING (partial or full)

GE is UNSTABLE.

Pivoting STABILIZES

GE! ☺

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well ill

alg

unstable stable

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$