Math 551 3/2

$\boxed{5763.txt}$

HW Submissions: From HW2 on ....

① You have a 4-digit HW Code (5763)

name of submitted File

$\underbrace{\text{HW}}_{\text{uppercase}}2\_5763.\underbrace{\text{pdf}}_{\text{lowercase}}$

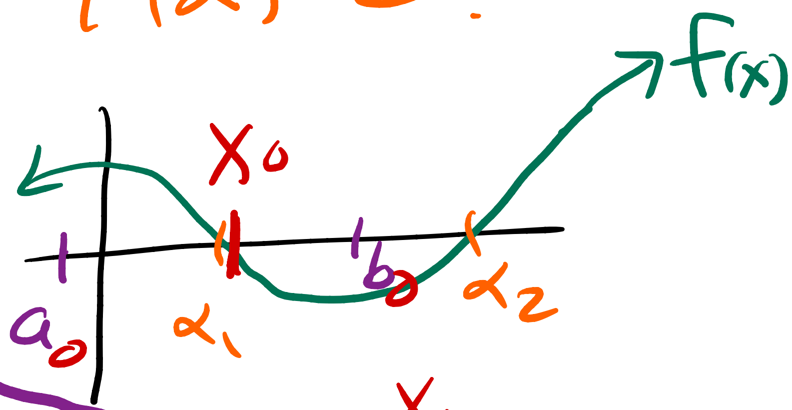
underscore

② HW #3 due 3/4 @9pm

③ I will email instructions on resubmitting HW#2

# Roots of Nonlinear Equations (root $\alpha$ or $x^*$)

Defn: $\alpha$ is a root of $f(x)$ if $f(\alpha)=0$.




Bisection: (Does not generalize $\mathbb{R}^n$ $n \geq 2$ ☹)

① $f \in C([a,b])$
② $f(a) \cdot f(b) < 0$

$\overset{IVT}{\Longrightarrow}$ $\alpha \in (a,b)$, $f(\alpha)=0$

$x_0$
$x_1$
$x_2$
$\vdots$
$\downarrow$
$\alpha$?

Error Estimate: $|\alpha - x_n| \leq \dfrac{b-a}{2^{n+1}}$

Given tolerance $n_{min} = \left\lceil \log_2\left(\dfrac{b-a}{tol}\right) - 1 \right\rceil$

Main Cost is 1 function eval per step

# Fixed-Point Methods

**Defn:** $\alpha$ is a FIXED-PT of $g(x)$ if $g(\alpha)=\alpha$.

Basic Idea: Root Problem $\xrightarrow{\text{Reformulate Problem}}$ Fixed-pt Problem

$\alpha$ is a root of $f(x)$ $(f(\alpha)=0)$ $\Longleftrightarrow$ $\alpha$ is a fixed point of $g(x)$ $(g(\alpha)=\alpha)$

## Fixed-Pt Iteration

Given $x_0$ (initial guess)

$$x_{n+1}=g(x_n)$$

$$n=0,1,2,3,\ldots$$

**Question**

$$\lim_{n\to\infty} x_n \stackrel{?}{=} \alpha$$

Ex: $F(x) = x^2 - 3$, $\alpha = \pm\sqrt{3}$

$g_1(x)$: $x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \frac{x^2}{x} = \frac{3}{x} = g_1(x)$

So $g_1(x) = \frac{3}{x}$ and $\underbrace{g_1(\pm\sqrt{3}) = \pm\sqrt{3}}$ fixed-pts

$g_2(x)$: $x^2 - 3 = 0 \Rightarrow \frac{x^2-3}{2} = \frac{0}{2} = 0 \Rightarrow x - \frac{x^2-3}{2} = x - 0 = x$

So $g_2(x) = x - \frac{x^2-3}{2} \Rightarrow \underbrace{g_2(\pm\sqrt{3}) = \pm\sqrt{3}}$ fixed-pts

$g_3(x)$: $x^2 - 3 = 0 \Rightarrow \frac{x^2-3}{2x} = 0 \Rightarrow x - \frac{x^2-3}{2x} = x - 0 = x$

So $g_3(x) = x - \frac{x^2-3}{2x} \Rightarrow g_3(\pm\sqrt{3}) = \underbrace{\pm\sqrt{3}}_{\text{fixed-pts}}$

$= \frac{2x^2 - x^2 + 3}{2x} = \frac{x^2+3}{2x} = \frac{1}{2}\left(x + \frac{3}{x}\right)$

3 operations

## Iterations

Suppose $x_0 = 1.5$

**Case 1:** $g_1(x) = \frac{3}{x}$

| $n$ | $x_n$ |
|---|---|
| 0 | 1.5 |
| 1 | 2 |
| 2 | 1.5 |
| 3 | 2 |

DIVERGES!

**Case 2:** $g_2(x) = x - \frac{x^2-3}{2}$

| $n$ | $x_n$ | $\lvert\sqrt{3}-x_n\rvert$ |
|---|---|---|
| 0 | 1.5 | 0.23205... |
| 1 | 1.875 | 0.14294... |
| 2 | 1.617... | 0.1148... |
| 3 | 1.8095... | 0.0774... |
| 4 | 1.6723... | 0.0597... |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $\infty$ | $\sqrt{3}$ | 0 |

CONVERGES!

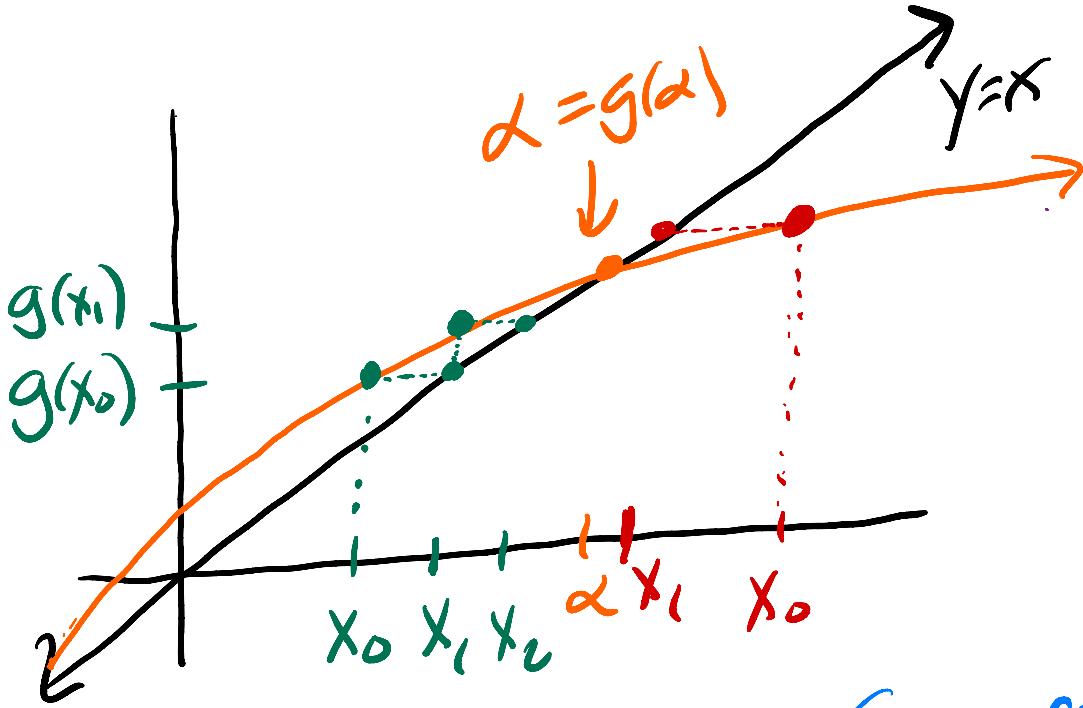
Newton's Method

**Case 3:** $g_3(x) = x - \frac{x^2-3}{2x}$

| $n$ | $x_n$ | $\lvert\sqrt{3}-x_n\rvert$ |
|---|---|---|
| 0 | 1.5 | 0.23205... |
| 1 | 1.75 | 0.01794... |
| 2 | 1.7321429... | 0.000092... |
| 3 | 1.7320509... | 0.0000001... |

CONVERGES RAPIDLY!

# Fixed-Pt Methods Graphically




**Given $x_0$**

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

appears to go to $\alpha$!

Converges

**Given $x_0$**

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

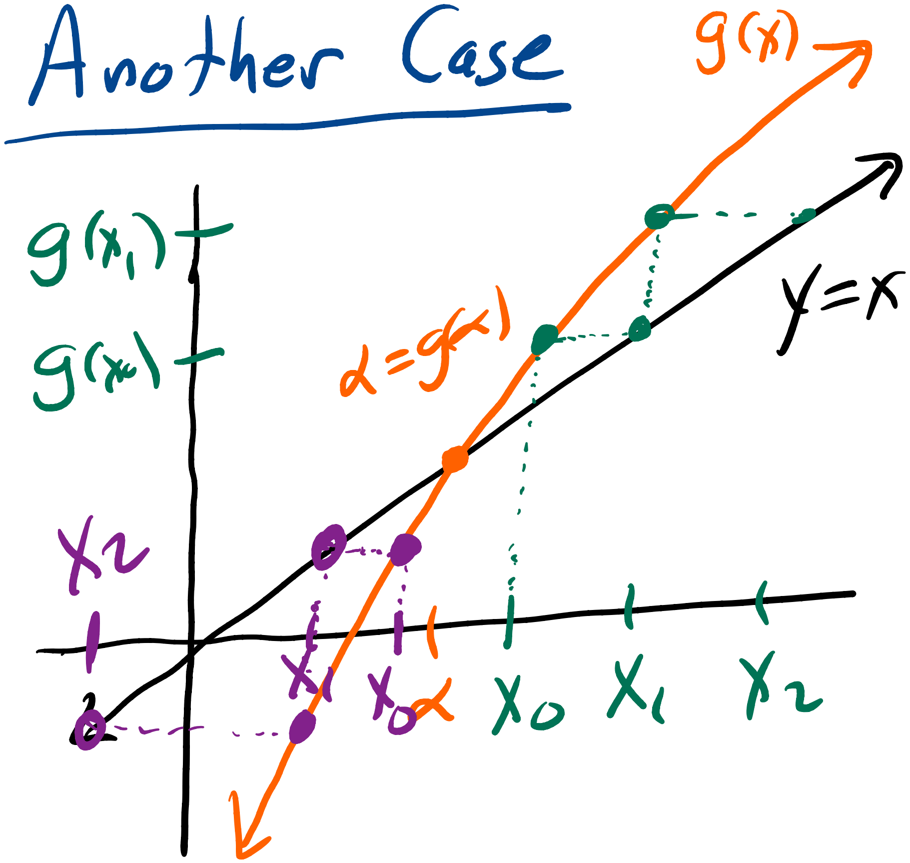
appears to go to $\alpha$!

$0 \le |g'(\alpha)| < 1$

**Note:**

$$|g'(\alpha)| < 1$$

# Another Case




Given $x_0$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

$\lim_{n \to \infty} x_n \neq \alpha$

Given $x_0$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

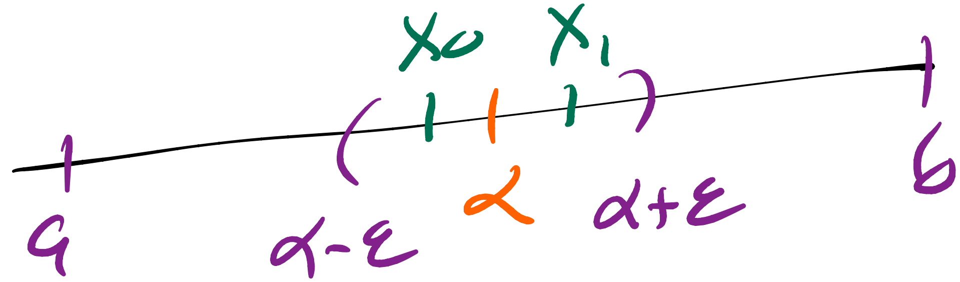
$\vdots$

$\lim_{n \to \infty} x_n \neq \alpha$

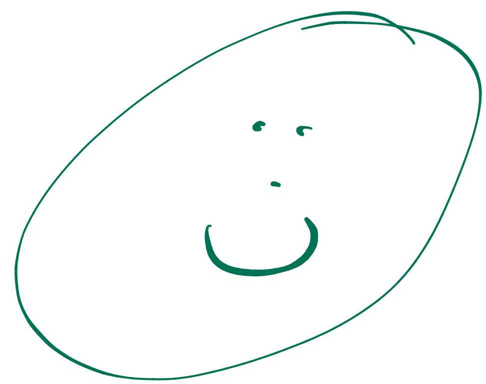
Note!

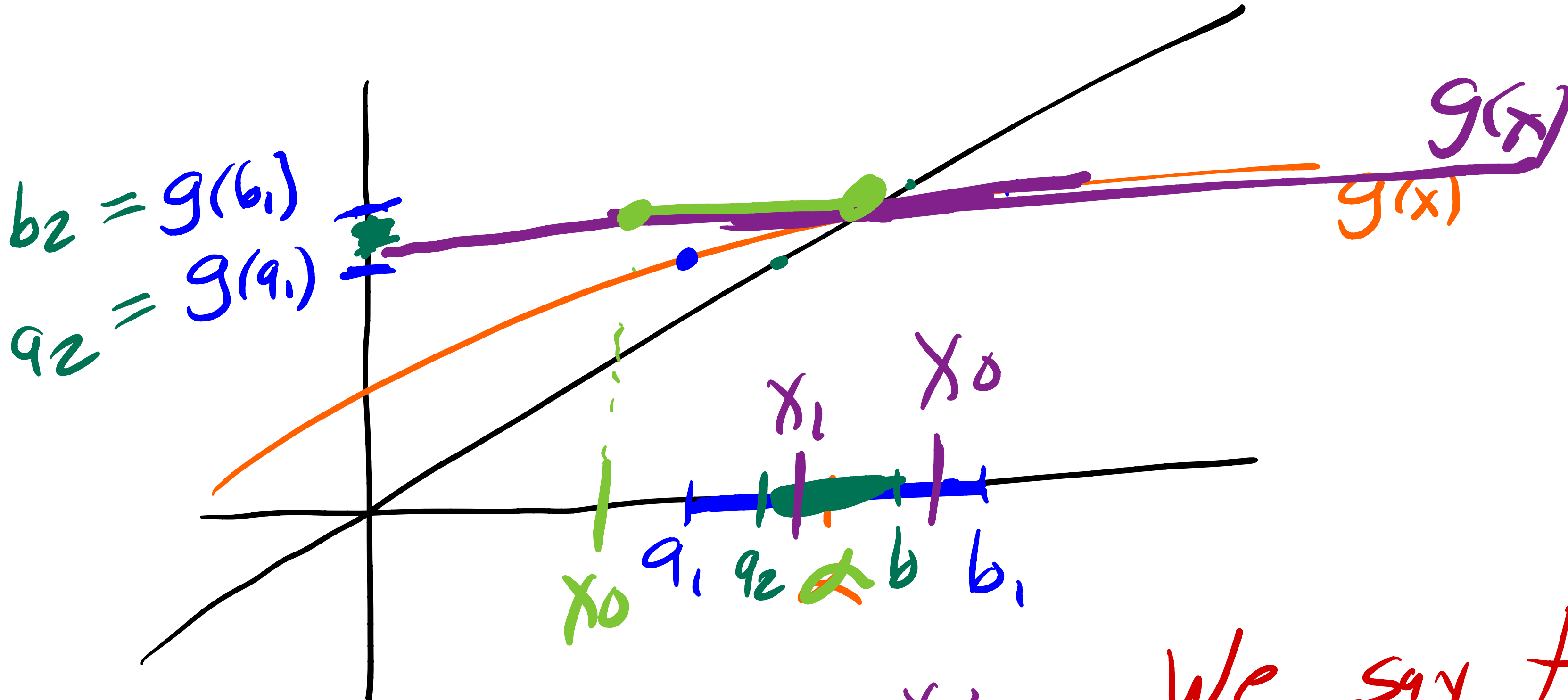
$|g'(\alpha)| \geq 1$

We will show if $g \in C^1([a,b])$ and $|g'(\alpha)| < 1$ then there exists $\varepsilon > 0$ such that if $x_0 \in (\alpha - \varepsilon, \alpha + \varepsilon)$




then $\lim_{n \to \infty} x_n = \alpha$






$\alpha \in [a_1, b_1]$ ($x_0$)

$\alpha \in [a_2, b_2]$ ($x_1$)

$\alpha \in [a_3, b_3]$ ($x_2$)

We say that near $\alpha$ $g(x)$ is a

CONTRACTION

# Recall:

Case 1: $g_1(x) = \frac{3}{x} \Rightarrow g_1'(x) = -\frac{3}{x^2}$

and $\left|g_1'(\sqrt{3})\right| = \left|\frac{-3}{\sqrt{3}^2}\right| = 1 \not< 1$

Case 2: $g_2(x) = x - \frac{x^2-3}{2} = -\frac{1}{2}x^2 + x + 3/2$

$g_2'(x) = -x+1 \Rightarrow \left|g_2'(\sqrt{3})\right| = \left|1-\sqrt{3}\right| \approx 0.7 < 1$

Case 3: $g_2(x) = x - \frac{x^2-3}{2x} = \frac{2x^2-x^2+3}{2x} = \frac{1}{2}x + \frac{3}{2}x^{-1}$

$g_2'(x) = \frac{1}{2} - \frac{3}{2}x^{-2} \Rightarrow \left|g_2'(\sqrt{3})\right| = \left|\frac{1}{2} - \frac{3}{2}\cdot\left(\frac{1}{\sqrt{3}}\right)^2\right|$

$=$