Math 551

3/25

- HW #5 due tomorrow @9PM
- HW #6 due 4/2 @9PM

Office Hours: M, W, Th

7PM - 8PM

GE : $Ax = b \quad A \in \mathbb{R}^{n\times n} \quad x, b \in \mathbb{R}^n$

Soln:

$$\left[A^{(1)} \,\vdots\, b^{(1)}\right] \xrightarrow{\text{row operation}} \left[A^{(2)} \,\vdots\, b^{(2)}\right] \xrightarrow{\text{row operations}} \cdots \rightarrow \left[A^{(n)} \,\vdots\, b^{(n)}\right]$$

where $A^{(n)} = U$ uppertriangular

To solve $Ax=b$ we solve $\boxed{Ux = b^{(n)}}$ using backsubstion, which costs $O(n^2)$

Ques: what is the cost of going from $\left[A^{(1)} \vdots b^{(1)}\right]$ to $\left[A^{(n)} \vdots b^{(n)}\right]$

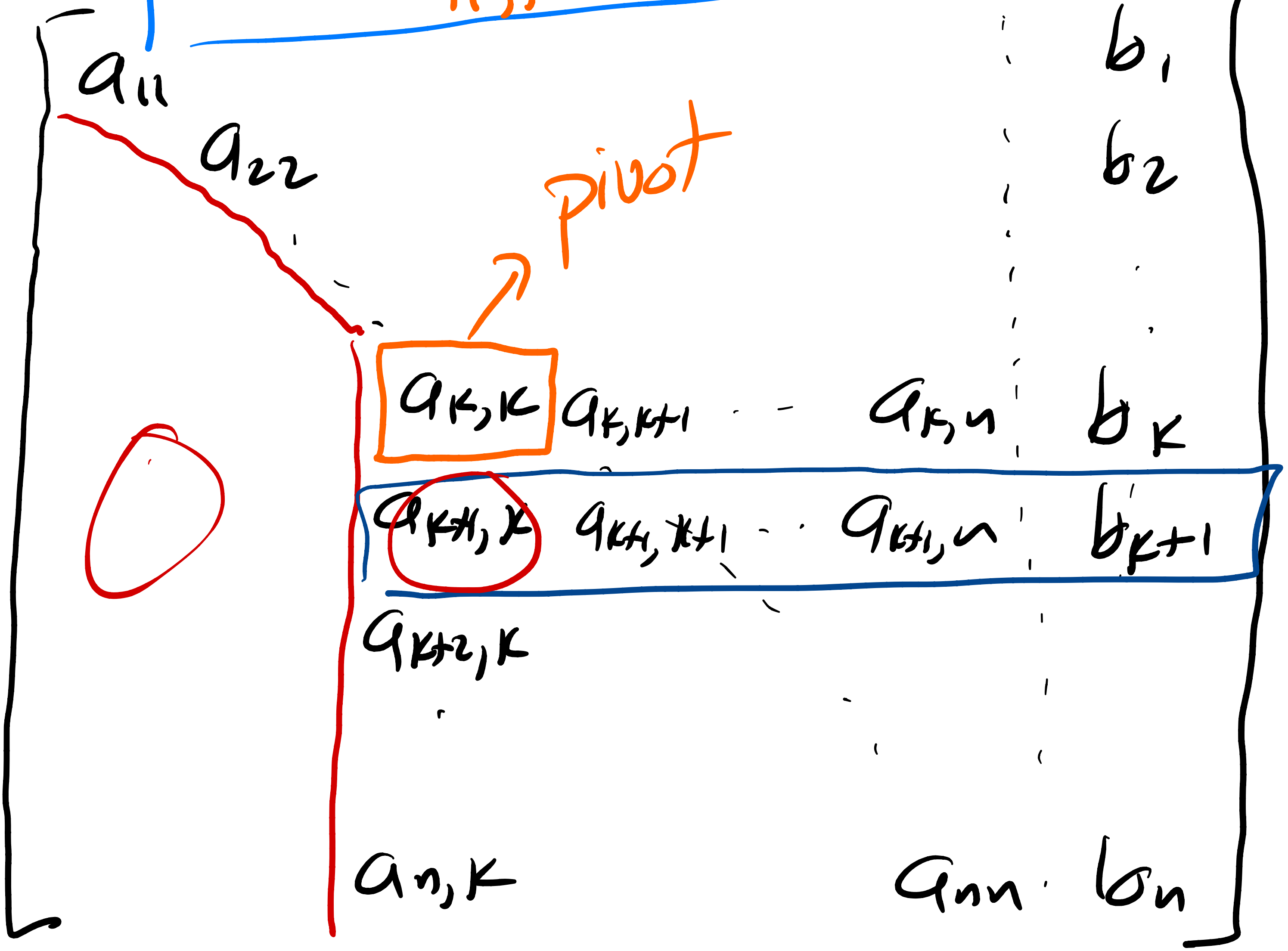
Cost of GE:

At the k^{th} step (or stage)

For $k = 1, 2, 3, \ldots, n-1$, we eliminate x_k from eqns $k+1, k+2, \ldots, n$

$$m_{k+1,k} = \frac{a_{k+1,k}}{a_{k,k}} \qquad R_{k+1} - m_{k+1,k} R_k \to R_{k+1}$$

$$[A^{(k)} \vdots b^{(k)}] =$$



$\downarrow$ Pseudo Code

$$[A^{(k+1)} \vdots b^{(k+1)}]$$

Pseudo-Code

At the k^{th} step

```
      For each row $i = k+1, \dots, n$
          $m_{i,k} = a_{i,k}/a_{k,k}$                       1 div
n-K       $a_{i,k} = 0$
          $b_i = b_i - m_{i,k} \cdot b_k$                   1 mult   1 sub
          for each column $j = k+1, \dots, n$
n-K           $a_{i,j} = a_{i,j} - m_{i,k} \cdot a_{k,j}$   1 mult   1 sub
          end
      end
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outer loop at the K^{th} step

+/- $(n-K)[1+(n-K)] = K^2-(2n+1)K+(n^2+n)$

*/÷ $(n-K)[2+(n-K)] = K^2-2(n+1)K+(n^2+2n)$

over $K=1,2,\ldots,n-1$ total cost is

+/- $$\sum_{K=1}^{n-1} K^2-(2n+1)K+(n^2+n) = \frac{1}{3}n^3+n^2-\frac{1}{3}n$$
$$= O\left(\frac{1}{3}n^3\right)$$

*/÷ $$\sum_{K=1}^{n-1} K^2-2(n+1)K+(n^2+n) = \frac{1}{3}n^3+\frac{3}{2}n^2-\frac{5}{6}n$$
$$= O\left(\frac{1}{3}n^3\right)$$

and we have

$$\sum_{K=1}^{n-1} K^2 - (2n+1)K + (n^2+n)\cdot 1$$

$$= \sum_{K=1}^{n-1} K^2 - (2n+1)\sum_{K=1}^{n-1} K + (n^2+n)\sum_{K=1}^{n-1} 1$$

$$= (n-1)(n)(2n-1) - (2n+1)\frac{(n-1)n}{2} + (n^2+n)(n-1)$$

$$= \frac{1}{3}n^3 + n^2 - \frac{1}{3}n$$

$$= O\left(\frac{1}{3}n^3\right)$$

$$\sum_{K=1}^{m} 1 = m \qquad \sum_{K=1}^{m} K = \frac{m(m+1)}{2} \qquad \sum_{K=1}^{m} K^2 = \frac{m(m+1)(2m+1)}{6}$$

Total cost to solve $Ax = b$

	$+/-$	$*/\div$	total
GE	$O(\frac{1}{3}n^3)$	$O(\frac{1}{3}n^3)$	$O(\frac{2}{3}n^3)$
Back sub	$O(\frac{1}{2}n^2)$	$O(\frac{1}{2}n^2)$	$O(n^2)$

$$\text{Total} = O(\tfrac{2}{3}n^3)$$

Recall, we talked about $A = \underset{\uparrow \atop \text{unit lower triangular}}{L}\ \underset{\uparrow \atop \text{upper triangular}}{U}$.

So what is L?

Matrix Formulation of GE

$A^{(k)}$ denotes the k^{th} derived matrix in GE before step k, and we can write

$$A^{(k+1)} = E_k A^{(k)}, \quad E_k = \begin{array}{ccccccc} & & k^{th} & & & & \\ 1 & & & & & & \\ & 1 & & & 0 & & \\ & & \ddots\ 1 & & & & k^{th} \\ 0 & & -m_{k+1,k} & 1 & & & \\ & & -m_{k+2,k} & & \ddots & & \\ & & \vdots & & & & \\ & & -m_{n,k} & & & 1 & \end{array}$$

Note:

① E_k is a unit lower triangular matrix

Elementary Matrix

② $\det(E_k) = 1 \Rightarrow E_k^{-1}$ exists

③ E_k results from apply $R_i - m_{i,k} R_k \rightarrow R_i$, $k+1 \leq i \leq n$ to I.

So given $A = A^{(1)}$ we have

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$A^{(2)} = E_1 A^{(1)} = E_1 A$$

$$A^{(3)} = E_2 A^{(2)} = E_2 E_1 A$$

$$\vdots$$

$$\underbrace{A^{(n)}}_{U} = E_{n-1} A^{(n-1)} = E_{n-1} E_{n-2} \cdots E_2 E_1 \cdot A$$

$$\Rightarrow \quad E_{n-1} E_{n-2} \cdots E_2 E_1 A = U$$

$$\Rightarrow \quad A = (E_{n-1} E_{n-2} \cdots E_2 E_1)^{-1} U$$

$$\Rightarrow \quad A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1}}_{L} \cdot U$$

or

$$A = L \cdot U$$

Note: $E_k^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ 0 & & m_{k+1,k} & \ddots & \\ & & m_{k+2,k} & & \\ & & \vdots & & \\ & & m_{n,k} & & 1 \end{bmatrix}$