Math 551 3/23

- HW 5 due 3/26 @ 9PM
- Office Hours
  M, W, Th 7PM - 8PM

Solving $Ax=b$ $\quad A \in \mathbb{R}^{n\times n} \quad x, b \in \mathbb{R}^n$

TFAE

(1) A is invertible, i.e., A^{-1} exists and $AA^{-1} = A^{-1}A = I$

(2) $Ax=b$ has a unique solution $x = A^{-1}b$

(3) $Ax=0$ has only the TRIVIAL soln $x=0$.

(4) $\det(A) \neq 0$ $\left(\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0 \right)$

(5) All eigenvalues, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, are non-zero

① Suppose $\lambda = 0$ is a e-val, $x \neq 0$ eigenvector

$$Ax = \lambda x = 0 \cdot x = 0$$

Ex: $n=2$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \det(A) = a_{1,1} \cdot a_{2,2} - a_{2,1} \cdot a_{1,2}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,2} \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = 1 \cdot 4 - 3 \cdot 2 = -2$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

check

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Strategy for solving $Ax=b$

① Find L and U such that $A = \overbrace{L \cdot U}^{\text{decomposition}}$

where $L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \ell_{2,1} & \ddots & & & 0 & \\ \vdots & & & & & \\ \ell_{n,1} & \cdots & \ell_{n,n-1} & 1 & & \end{bmatrix}$, $U = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ & \ddots & & \\ & 0 & & u_{n,n} \end{bmatrix}$

② Solve $Ax=b \iff L\overbrace{Ux}^{z}=b$

(a) Solve $Lz=b$ by FORWARD Substitution

(b) Solve $Ux=z$ by BACK Substitution

$$0 \neq \det(A) = \det(LU) = \underbrace{\det(L)}_{\neq 0} \cdot \underbrace{\det(U)}_{\neq 0}$$

Cost? In terms of # of +/- and */÷

(1) Cost of determining L and U?

ans: We will show the cost is $O(cn^3)$

(2) Cost of solving (1) $Lz = b$ (2) $Ux = z$

ans: We will show the cost for each is $O(cn^2)$

Ex: $A = U$, solve $Ux = b$ upper triangular

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

Back substitution

$$a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

$$\ddots$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

$$\boxed{a_{n,n}x_n = b_n} \Rightarrow x_n = \frac{b_n}{a_{n,n}}$$

$$\det(A) = \prod_{i=1}^{n} a_{i,i} \neq 0 \Rightarrow a_{i,i} \neq 0 \quad 1 \le i \le n$$

$$x_n = b_n / a_{n,n}$$

$$x_{n-1} = (b_{n-1} - a_{n-1,n} x_n) / a_{n-1,n-1}$$

$$x_{n-2} = \frac{(b_{n-2} - a_{n-2,n} x_n - a_{n-2,n-1} x_{n-1})}{a_{n-2,n-2}}$$

$$\vdots$$

$$x_1 = \frac{\left(b_1 - \sum_{i=2}^{n} a_{1,i} x_i\right)}{a_{1,1}}$$

$$\begin{array}{cc} */\div & +/- \\ \hline \end{array}$$

$$\left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ & \\ n & n-1 \end{array}\right]$$

$$+$$

$$\begin{array}{cc} \hline ? & ? \end{array}$$

Gauss: $\sum_{i=1}^{m} i = 1+2+3+\cdots+m = \frac{m(m+1)}{2}$

Operation Counts

$*/\div \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n = O(\frac{1}{2}n^2)$

$+/- \quad \sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n = O(\frac{1}{2}n^2)$

Total Cost $\qquad O(n^2)$

Note: Cost to solve $Lz = b$ $O(n^2)$

$Ux = z$ $O(n^2)$

$O(2n^2)$

Gauss-Jordan reduction

235: $\underbrace{[A \,\vdots\, b]}_{\text{augmented matrix}} \xrightarrow{\text{Row Operations}} [I \,\vdots\, x = A^{-1}b]$

551: $[A \,\vdots\, b] \xrightarrow{\text{Row Operations}}$ (1) $[U \,\vdots\, \tilde{b}]$

(2) Solve $Ux = \tilde{b}$

Fact: We get L for Free!

Elementary Row Operations

① multiply any row by a non-zero constant $\qquad cR_i \to R_i \quad (c \neq 0)$

② add a multiple of one row to another $\qquad cR_i + R_j \to R_j \quad (\text{any } c)$

③ interchange two rows $\qquad R_i \leftrightarrow R_j$

Each leaves the solution X unchanged!

Ex:

$$2x_1 + x_2 + 3x_3 = 1$$
$$4x_1 + 4x_2 + 7x_3 = 1$$
$$2x_1 + 5x_2 + 9x_3 = 3$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad L \qquad\qquad U$$

Augmented System

$$\left[\begin{array}{ccc:c} \boxed{2} & 1 & 3 & 1 \\ 4 & 4 & 7 & 1 \\ 2 & 5 & 9 & 3 \end{array}\right]$$

$$R_2 - \underbrace{2}_{\text{multiplier}} R_1 \to R_2 \qquad \left(2 = \frac{4}{2} = \frac{a_{2,1}}{a_{1,1}}\right)$$

pivot

$$R_3 - \underbrace{1}_{\text{multiplier}} R_1 \to R_3 \qquad \left(1 = \frac{2}{2} = \frac{a_{3,1}}{a_{1,1}}\right)$$

$$\left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & \boxed{2} & 1 & -1 \\ 0 & 4 & 6 & 2 \end{array}\right]$$

$$R_3 - \underbrace{2}_{\text{multiplier}} R_2 \to R_3 \qquad \left(2 = \frac{4}{2} = \frac{a_{3,2}}{a_{2,2}}\right)$$

pivot

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

Back substitution!

$$\Rightarrow \quad 4x_3 = 4 \quad \Rightarrow \quad x_3 = 1$$

$$\Rightarrow \quad x_2 = (-1 - x_3)/2 = {-2}/{2} = -1$$

$$\Rightarrow \quad x_1 = (1 - x_2 - 3x_3)/2$$

$$= (1 - (-1) - 3)/2 = -\frac{1}{2}$$

$$\Rightarrow \quad x = \left[-\tfrac{1}{2} \;\; -1 \;\; 1\right]^T$$