Math 551 3/18

- HW # 4 due tomorrow @ 9PM
- HW #5 3/26 @9PM

Ax=b: Read $\underbrace{5.1}_{\text{GE}}$ and $\underbrace{5.2}_{\text{LU}}$

Chapter 4 is a Linear Algebra Primer

Newton's Method

Find a root α of $f(x)$.

Algorithm: Given x_0, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= g(x_n)$

$n = 0, 1, 2 \ldots$

where $g(x) = x - \frac{f(x)}{f'(x)}$ } Newton Iteration Function

Thm: $f \in C^2([a,b])$, α is a SIMPLE ROOT of $f(x)$ ($f'(\alpha) \neq 0$), then if x_0 is chosen sufficiently close to α, the Newton's mthd converges with order of convergence is AT LEAST 2, i.e. $p = 2$. (quadratic)

Newton's Method for 2 Dimensions

$F: \mathbb{R}^2 \to \mathbb{R}^2 \qquad F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$

Find $(\alpha, \beta)^T \in \mathbb{R}^2$ such that

$$F(\alpha,\beta) = 0 \quad \text{or} \quad \begin{matrix} f_1(\alpha,\beta) = 0 \\ f_2(\alpha,\beta) = 0 \end{matrix}$$

Recall: 1-dimension

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - [f'(x_n)]^{-1} \cdot f(x_n)$$

n-dimensions:

$$\vec{x}_{n+1} = \vec{x}_n - [\underbrace{Df(\vec{x}_n)}_{\text{Jacobian}}]^{-1} F(\vec{x}_n)$$

$\underline{n=2}$:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_n, y_n) & \frac{\partial f_1}{\partial y}(x_n, y_n) \\ \frac{\partial f_2}{\partial x}(x_n, y_n) & \frac{\partial f_2}{\partial y}(x_n, y_n) \end{bmatrix}^{-1} \begin{bmatrix} F_1(x_n, y_n) \\ F_2(x_n, y_n) \end{bmatrix}$$

or with $\hat{X}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ we can write

$$\hat{X}_{n+1} = \hat{X}_n - [\underbrace{DF(\hat{X}_n)}_{\substack{\text{Jacobian} \\ n \times n \text{ matrix} \\ \text{of } 1^{st} \text{ partials}}}]^{-1} F(\hat{X}_n)$$

Solving $Ax = b$ n eqns in n unknowns

$$a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n = b_1$$

$$a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n = b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n,1} x_1 + a_{n,2} x_2 + \cdots + a_{n,n} x_n = b_n$$

or

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for } 1 \le i \le n$$

or in matrix form

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}\end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}}_{b}$$

square

$A \quad (n \times n) \qquad x \quad (n \times 1) \qquad b \quad (n \times 1)$

or $Ax = b$

We will *always* assume that A is INVERTIBLE!

The following are <u>EQUIVALENT</u> (TFAE)

$A \in \mathbb{R}^{n \times n}$

(1) A invertible, i.e. A^{-1} exists and $AA^{-1} = A^{-1}A = I$ ($n \times n$ identity mtx)

(2) $Ax = b$ has a unique soln $x = A^{-1}b$ for each b.

(3) $Ax = 0$ has only the TRIVIAL soln $x = 0$,

(4) $\det(A) \neq 0$ $\boxed{\det(A) = \prod_{i=1}^{n} \lambda_i}$,

(5) The eigenvalues of A, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, are all non-zero!

$$\underline{A}x = \lambda x$$

$$\frac{d^2}{dx^2}(\sin x) = -\sin x$$

$$\frac{d^2}{dx^2}(\cos x) = -1 \cdot \cos x$$

$$\frac{d^2}{dx^2}(e^x) = 1\,(e^x)$$